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DIPARTIMENTO DI SCIENZE PER L'ECONOMIA E L'IMPRESA

Working Papers - Economics

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Working Paper N. 16/2019

DISEI, Università degli Studi di Firenze
Via delle Pandette 9, 50127 Firenze (Italia) www.disei.unifi.it

On individual incentives to bundle in oligopoly*

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April 24, 2019

Abstract

This paper examines competition in an oligopoly with multiproduct firms when some firms bundle but other firms sell their products separately, whereas the existing literature on competitive bundling focuses on the extreme cases of competition among bundles or among individual products. Our analysis reveals each firm's individual incentive to bundle, and allows to study a two-stage game in which first each firm chooses its pricing strategy (bundling or independent pricing), then price competition occurs given the price regime each firm has selected at stage one. When firms are ex ante symmetric, we find that bundling is weakly dominated by independent pricing. In a setting in which a firm's products have higher quality than its rivals' products, individual incentives to bundle emerge (eventually for all firms) if the quality difference is large.

Keywords: Oligopoly, Pure bundling, Independent Pricing, Competitive bundling; **JEL Codes**: D43, L13.

*This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. Declarations of interest: none

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1 Introduction

The use of bundling under monopoly is relatively well understood,[1] but perhaps the same cannot be said for bundling under oligopoly. There exists a literature (see below in this introduction) that studies competition among multiproduct firms when each firm offers its products in a bundle, and compares the outcome with the outcome of competition when no firm bundles.[2] This allows to identify settings in which firms' profits are higher/lower if competition occurs under bundling/no bundling. However, to the best of our knowledge there exists no study of competition among more than two firms given that some firms bundle and some do not. This paper (partially) fills this gap. The results we obtain allow to analyze a two-stage game in which first each firm decides whether to offer its products separately or in a bundle, then price competition takes place. Our results suggest that among symmetric firms bundling is unprofitable for a firm, independently of its rivals' choices. We also consider a setting with a dominant firm which offers products of higher quality with respect to the other firms' products. We show that as the quality difference increases, incentives to bundle arise first for the dominant firm alone, then also for the other firms.

In detail, we consider an oligopoly setting with $n \geq 3$ symmetric firms, each offering two products A and B, and each consumer needs one unit of product A and one unit of product B. The products of different firms are differentiated. The market for each product is represented by a Salop's circle [Salop (1979)] in which firms are equidistantly located and consumers are uniformly distributed; firms are located in the same way on the two circles. In this context we inquire the incentives of each single firm to bundle by examining the following two-stage game:

- at stage one (the marketing stage), each firm chooses between independent sales and bundling;
- at stage two (the price competition stage), firms compete by choosing prices given the price regime each firm has selected at stage one.

This timing is sensible in a situation in which a firm's choice in stage one between bundling and independent pricing is irreversible, perhaps because of technical reasons involving the way the firms' products are designed.[3] One difficulty in our analysis is that when some firms bundle and some do not, we need to study an asymmetric price competition game for which we have to apply a case-by-case approach that depends on the total number n of firms, on the number of firms which bundle, and on the locations of these firms.[4] This forces us to restrict our analysis to $n = 3$ or $n = 4$.

For these cases we find that independent sales weakly dominate bundling for each firm. Thus, no firm has any incentive to bundle, regardless of the marketing choices of the other firms.[5] In particular, if we fix a firm i and assume that all other firms choose independent sales, then firm i prefers independent sales to bundling. In order to see why, suppose that firm i bundles. Then it is impossible for consumers to "mix and match" one product of firm i and one product of another firm, and the revenue of firm i comes only from the

[1] See for instance Adams and Yellen (1976), McAfee, McMillan, and Whinston (1989), Fang and Norman (2006) and the references therein.

[2] There also exists a literature which investigates the use of bundling as an instrument to deter the entry of competitors: See for instance Whinston (1990) and Nalebuff (2004).

[3] Conversely, if a firm could costlessly and quickly switch from one price regime to the other, then it may be adequate to merge the two stages, assuming that at a single stage each firm chooses bundling (and the price for the bundle) or independent sales (and the prices for the individual products).

[4] For instance, if $n = 4$ and only one firm bundles, then the three firms that do not bundle are not in a symmetric position because one of them is not adjacent to the bundling firm: See Subsection 3.3.2.

[5] Since each consumer wants one unit of product A and one unit of product B, a firm's decision to bundle or not is irrelevant in case that all other firms bundle. Hence, there also exists an equilibrium in which all firms bundle, but that requires that all firms play a weakly dominated action.

sales of its bundle. As a starting point, assume that the price of the bundle of firm i is the sum of the prices of firm i's individual products in the equilibrium under independent sales. Then consider the mix and match consumers of firm i that under independent sales buy only one product of firm i. These consumers must decide whether to buy firm i's bundle or not.[6] It turns out that only a minority of them, those located not far away from firm i on both circles, buy the bundle. Thus firm i loses most of its mix and match consumers, which reduces its market share and profit: bundling reduces the options offered by firm i to consumers in a way that affects adversely firm i. Of course, bundling also affects the pricing incentives. In particular the reduced market share makes a price cut more profitable for firm i. Since prices are strategic complements, this leads to a price reduction also by the other firms, so that price competition is more intense. At the equilibrium prices, firm i recovers its original market share but with a substantially reduced price that makes its profit smaller than in case of competition among individual products.

We also examine an asymmetric setting with three firms in which one firm offers products with higher quality with respect to its competitors' products; we say that this firm is *dominant* over its rivals, and use a parameter $\alpha > 0$ to represent the quality difference.[7] In this setting, incentives to bundle emerge for the dominant firm if α is larger than a threshold α'. Precisely, when α is sufficiently large, bundling only by the dominant firm induces a majority of its mix and match consumers under independent sales to buy its bundle. Indeed, when facing the alternatives to buy both products, or no product at all, of the dominant firm, even a consumer located far away from it in one circle prefers the first alternative. Moreover, the value of α also affects the intensity of price competition under bundling: a large α makes the demand of the bundling firm less elastic, which softens price competition and increases the profit of all firms with respect to independent sales.

When only the dominant firm bundles, competition occurs between the dominant firm's bundle and the individual products of the dominated firms. However, there exists an α'' larger than α' such also a dominated firm wants to bundle (given that the dominant firm bundles) when $\alpha > \alpha''$. Then competition occurs among pure bundles, and the dominant firm faces less competition with respect to the case in which the dominated firms do not bundle, as consumers cannot mix and match the products of the dominated firms. This reduced competition increases the demand for the dominant firm, but also induces it to be less aggressive in pricing. This latter effect benefits the dominated firms and ultimately, from their viewpoint, dominates the initial effect of demand loss, increasing their profits if $\alpha > \alpha''$. Therefore all firms bundle when $\alpha > \alpha''$.

As we mentioned above, the literature on competitive bundling has examined the cases in which all firms bundle their products. For instance, Matutes and Regibeau (1988) show that in a bidimensional Hotelling duopoly, competition between bundles is fiercer than competition under independent sales and thus lowers firms' profits.[8] However, more recent research shows that this result may not hold when more than two firms compete. Precisely, in the random utility model of Perloff and Salop (1985), Zhou (2017) shows that under suitable assumptions on the distribution of consumers' valuations, bundling essentially reduces the heterogeneity in consumer valuation. In particular, the density of the average per-product value has thinner tails compared to the density of the original single-product valuation. If the number of competing firms is sufficiently large, then thinner tails lead to higher profits for competition under bundling than competition under separate sales. Kim and Choi (2015) use a spatial model in which the market for each product is represented by a Salop's circle and compare competition among bundles with competition among individual products. They find that if there are at least four firms, then there exists a way to symmetrically locate the

[6] The other consumers do not alter their purchases because of the bundling of firm i.

[7] Hurkens et al. (2018) examine a similar setting with two firms. But in that setting, as soon as a firm bundles, competition occurs among bundles (see footnote 5). Hence, it cannot occur, in practice, that some firms bundle and some do not.

[8] See also Economides (1989) and Nalebuff (2000). Hurkens *et al.* (2018) obtain different results assuming that firms are asymmetric.

firms in the two circles (but not in the same way on the two circles) such that bundling generates higher profits than independent sales.

Whereas the above papers focus on the extreme cases of competition among individual products and competition among bundles, our paper examines competition when some firms offer their products in a bundle and other firms do not. We model product differentiation as in Kim and Choi (2015). For symmetric firms we know from Kim and Choi (2015) that if all firms are located in the same way on the two circles, then competition among bundles reduces each firm's profit with respect to competition among individual products; thus firms have no *collective* incentive to bundle. Our results establish that also no single firm has an *individual* incentive to bundle, regardless of the actions of the other firms. Conversely, in the setting with asymmetric firms a significant asymmetry generates incentives to bundle first for the dominant firm, and for all firms if the asymmetry is sufficiently strong.

The rest of this paper is organized as follows: Section 2 introduces the model; Section 3 deals with the price competition stage; Section 4 is about the marketing stage; Section 5 analyzes the asymmetric setting; Section 6 contains a few suggestions for future research. The appendix includes the proofs of our results, but since some parts of the proofs are long and not very instructive, they are left to the supplementary material.

2 The setting

We consider competition among $n \geq 3$ symmetric firms, denoted firm 1, firm 2, ..., firm n, each offering two different products, A and B. We use A_i (B_i) to denote product A (product B) offered by firm i, for $i = 1, ..., n$. Each consumer has a unit demand for product A and a unit demand for product B.

The firms offer differentiated products and we represent product differentiation using a spatial model in which each firm is located on two Salop's circles (Salop (1979)). More precisely, like in Kim and Choi (2014, 2015) the market for each product is represented by a circle with unit length, in which a point is denoted "origin". Each point on the circle is identified by a number $x \in [0, 1)$ which represents the distance between the origin and that point, moving clockwise from the origin.

A point's coordinate
on the circle
(Figure 1)
Distance between two
points x and y on the circle
(Figure 2)

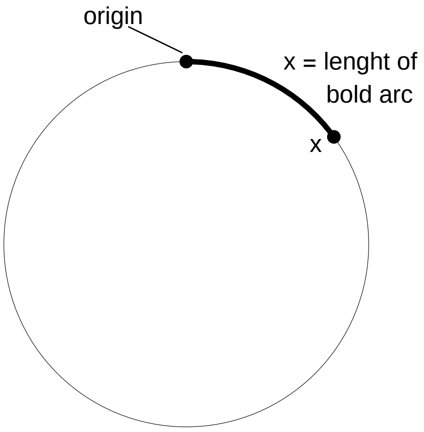


Figure 1

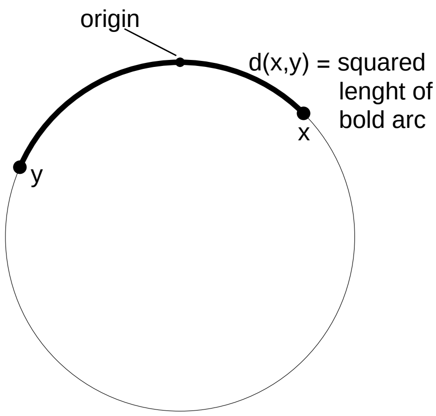


Figure 2

Each firm i is located at a point x_A^i on the circle for product A and at a point x_B^i on the circle for product B such that $x_A^i = x_B^i$. We consider symmetric firms, hence we assume that on each circle firms are equally-spaced (see for instance Figures 3 and 10 below). There is a unit mass of consumers and each consumer has a location x_A on the circle for product A and a location x_B on the circle for product B. These locations represent the consumer's ideal versions of the two products. For a consumer with locations x_A, x_B that buys product A from firm i, product B from firm j (notice that i may be equal to j), the utility is given by

$$V - d(x_A, x_A^i) + V - d(x_B, x_B^j) - \text{total consumer's payment to firms } i \text{ and } j \tag{1}$$

in which $V > 0$ represents the consumer's gross utility from consuming his preferred version of product A (of product B). With $d(x, y)$ we denote the quadratic distance between two generic points x and y on the circle: $d(x, y) = d(y, x)$ and for any x, y such that $0 \leq x \leq y < 1$ (without loss of generality) we have

$$d(x,y) = \begin{cases} (y-x)^2 & \text{if } 0 \leq y - x < \frac{1}{2} \\ (1-y+x)^2 & \text{if } \frac{1}{2} \leq y - x < 1 \end{cases} \tag{2}$$

Hence, $d(x, y)$ is the quadratic length of the shortest path that connects x to y on the circle. This sometimes requires to move clockwise, sometimes counterclockwise as the shortest path may pass through the origin: see Figure 2.

The term $d(x_A, x_A^i)$ in (1) is the distance between x_A and x_A^i on the circle for product A and it represents the reduction in the consumer's utility from consuming a version of product A which is different from his ideal one. A similar interpretation applies to $d(x_B, x_B^j)$.[9]

We assume that consumers' locations are independently and uniformly distributed on the two circles, and that V is high enough to make each consumer buy one product A and one product B in equilibrium. Therefore, A and B can be seen not only as products that can be independently consumed, but also as perfect complements, that is components of a system such that a consumer obtains a positive gross utility ($2V$) from consumption only if he consumes both a unit of A and a unit of B; for instance, a mobile phone and a battery charger for mobile phones. Consistently with this interpretation, we will often refer to products A_i, B_j as to "system ij", sometimes denoted with S_{ij}.

In this context we study each firm's incentives to offer its products separately or in a bundle. When firm i offers its products separately, we say that firm i practices *independent pricing* (IP). Thus it sets a price for its product A_i and a price for its product B_i, but since $x_A^j = x_B^j$ for $j = 1, ..., n$, we focus on the case in which firm i sets the same price p_i for both its products. The alternative to IP is that firm i practices *pure bundling* (PB), that is it offers its products jointly and sets a price P_i for the bundle that includes A_i and B_i (i.e., S_{ii}). From the point of view of a consumer, if firm i chooses IP then the consumer can combine (mix and match) product A_i (B_i) with product B_j (A_j) as long as also firm j has chosen IP. Conversely, if firm i chooses PB then a consumer either buys the bundle of firm i, or buys no product at all from firm i.[10]

After the firms' moves, each consumer faces a set of available alternatives to buy one unit of product A and one unit of product B. For instance, if there are three firms and firm 2 has chosen PB but firms 1 and 3 have both chosen IP, then $S_{11}, S_{22}, S_{13}, S_{31}, S_{33}$ are the available systems, whereas S_{12} (for instance) is not available. Among the feasible systems, each consumer chooses the one that yields the highest utility as

[9] We may multiply $d(x_A, x_A^i)$ and $d(x_B, x_B^j)$ by a positive number $t \neq 1$ to represent the importance for a consumer of consuming a product different from his ideal version, but that would have a multiplicative effect on profits and would not change our results qualitatively.

[10] When the products are viewed as perfect complements, it is possible to interpret the choice of IP or PB in terms of compatibility or incompatibility between products of different firms. Compatibility means that a consumer may combine products of different firms (i.e., A_i, B_j with $i \neq j$) without penalties in terms of a smaller utility. Incompatibility means that combining products of different firms yields a system that does not work. Then, if firm i chooses IP it means i offers products made according to an industry-wide standard which are compatible with the products of another firm j if also j has chosen IP. If instead firm i chooses PB, then its products are incompatible with the products of each other firm. Under compatibility, each battery charger can be used to recharge each mobile phone. Under incompatibility, a mobile phone can be recharged only by its specific battery charger and a consumer is forced to buy both the phone and the battery charger from the same firm.

evaluated in (1). That is equivalent to choosing the system with the lowest total cost given that $2V$ is the same for each system. For a consumer located at (x_A, x_B), the total cost of S_{ij} is

$$d(x_A, x_A^i) + d(x_B, x_B^j) + \text{total payment to buy } S_{ij} \tag{3}$$

For each firm i, let c denote the marginal cost for product A_i and for product B_i. We assume that c is large enough such that each consumer buys only one unit of product A, and only one unit of product B.[11] Then, since marginal costs have an additive effect on prices, without loss of generality we simplify the notation by setting $c = 0$ and interpret prices as profit margins.

Summarizing, we study a game with the following timing:

- Stage one: Each firm chooses whether it offers its products separately (IP) or in a bundle (PB); we call this stage "the marketing stage".
- Stage two: Each firm simultaneously sets the prices of its single products or the price of its bundle; we call this stage "the price competition stage".
- After stage two, consumers make their purchases as we have described above.

We denote the whole game with Γ. To it we apply the notion of Subgame Perfect Nash Equilibrium (SPNE), which requires to determine a Nash Equilibrium (NE) for each subgame of Γ that may be entered at stage two, that is for each possible price competition stage. Next section is devoted to this analysis.

3 The price competition stage

In this section we examine stage two in Γ, in which firms compete in prices given the price regimes determined at stage one. Precisely, we determine the equilibrium prices for each possible combination of price regimes (i.e., for each stage two subgame).

In order to distinguish different subgames at the price competition stage, we let $N \equiv \{1, 2, ..., n\}$ denote the set of all firms; N' is a generic subset of N. Then we use $\gamma^{N'}$ to denote the stage two subgame which is entered after at stage one each firm in N' has chosen PB and each firm in $N \backslash N'$ has chosen IP. Hence, $\gamma^{\varnothing}$ is the subgame which is played after each firm has chosen IP; $\gamma^{\{j\}}$ (or, more quickly, γ^j) is the subgame which is entered after only firm j has chosen PB; ...; γ^N is the subgame played after all firms have chosen PB. It is important to note that if $N' = N \backslash \{i\}$ (for an arbitrary i in N), then $\gamma^{N'}$ is equivalent to γ^N. Indeed, given that all the $n-1$ firms in N' have chosen PB, a consumer will buy the bundle of one firm in N', or the two products of firm i; in each case he will buy both products from a same firm, as in γ^N. Hence, both in $\gamma^{N'}$ and in γ^N competition occurs among the n bundles.

3.1 Competition under independent sales by all firms: Subgame $\gamma^{\varnothing}$

Here we consider the subgame $\gamma^{\varnothing}$ that is entered if each firm chooses IP at stage one. Given that each consumer's utility function is separable in the net utility obtained from the two products (see (1)), competition for product A is independent of competition for product B. Then it is immediate to identify a NE for each single market, given that firms are equidistantly located: See Proposition 1 in Kim and Choi (2015).

Lemma 1 *In subgame $\gamma^{\varnothing}$ there exists a symmetric NE such that in the market for product A (B) the price of each product is $\frac{1}{n^2}$. For each firm the equilibrium profit in each market is $\frac{1}{n^3}$ and the total profit is $\frac{2}{n^3}$.*

[11] Otherwise, in principle a consumer may buy the bundle of a firm i and then may want to buy a product B_j (for some firm $j \neq i$ which offers its products separately) because his location for product B is closer to x_B^j than to x_B^i. If c is sufficiently large, then the price of product B_j will be large enough to discourage this purchase.

3.2 Competition among bundles: Subgame γ^N, or $\gamma^{N\backslash\{i\}}$ for $i = 1, ..., n$

Here we consider the case in which at least $n-1$ firms have chosen PB at stage one. Then the subgame γ^N (or an equivalent one) is entered. Given that $x_A^i = x_B^i$ for $i = 1, ..., n$ and that firms are equidistantly located, Kim and Choi (2014) determine the following symmetric NE.

Lemma 2 (Kim and Choi (2014)) *In subgame γ^N there exists a symmetric NE such that the price for the bundle of each firm, denoted P^N, is*

$$P^N = \begin{cases} \frac{2}{n^2+3} & \text{if } n \text{ is odd} \\ \frac{2}{n^2+4} & \text{if } n \text{ is even} \end{cases}$$

The equilibrium profit for each firm is $\frac{1}{n}P^N$.

From Lemmas 1 and 2 it is immediate to see that each firm's profit is greater in $\gamma^{\varnothing}$ than in γ^N, since $\frac{2}{n^3} > \frac{1}{n}P^N$ for each $n \geq 3$. Therefore, all firms prefer that competition takes place among individual products rather than among bundles.[12]

3.3 Competition among bundles and individual products: Asymmetric subgames

In this subsection we examine subgames of Γ in which at least one firm offers its products in a bundle, but no more than $n-2$ firms do so. We call these *asymmetric* subgames because in these subgames firms are not in a symmetric situation. Precisely, a firm that has chosen PB at stage one is obviously in a different situation with respect to a firm that has chosen IP, but actually even firms which have made the same stage one choice may be asymmetric at stage two. For instance, consider subgame γ^3 when $n = 4$ and refer to Figure 10 below. Firms 1, 2, 4 have chosen IP and firms 2 and 4 are symmetric, but 1 and 2 are not so because one firm adjacent to firm 2 bundles (i.e., firm 3), whereas no firm adjacent to firm 1 bundles; thus 1 does not directly compete with a bundling firm, unlike 2. Asymmetric subgames are more complicated to study than $\gamma^{\varnothing}$ and γ^N, and this prevents us from providing results which hold for each n (Lemmas 1-2 do so for subgames involving symmetric firms). Rather, we focus on the asymmetric subgames of Γ when $n = 3$ or $n = 4$.[13]

3.3.1 Asymmetric subgames when $n = 3$

When there are three firms, the only asymmetric subgames are $\gamma^1, \gamma^2, \gamma^3$, which are entered after a single firm has chosen PB.

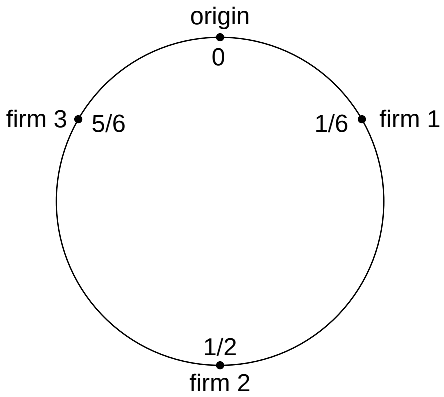
To fix the ideas, here we examine γ^2, that is we suppose that only firm 2 has chosen PB, but the results we obtain apply also if firm 1 or firm 3 is the unique bundling firm. In γ^2 it is computationally convenient (but implies no loss of generality) to assume that in both circles firm 1 is located at $\frac{1}{6}$, firm 2 is located at

[12] Kim and Choi (2015) show that if $n \geq 4$, then relaxing the assumption $x_A^i = x_B^i$ for $i = 1, ...n$, while still keeping firms symmetrically located in each circle, affects the equilibrium prices and profits in γ^N and in some cases leads to higher profits in γ^N than in $\gamma^{\varnothing}$.

[13] If $n = 2$, then γ^1 and γ^2 are equivalent to γ^{12}; hence there are no asymmetric subgames and Lemmas 1-2 cover all the subgames of Γ. Assuming $n \geq 3$ rules out this trivial case.

$\frac{1}{2}$, firm 3 is located at $\frac{5}{6}$:[14] see Figure 3:

Figure 3
Distribution of firms when $n = 3$:
over the circle for each product



We denote with p_1 (p_3) the price firm 1 (firm 3) charges on each of its products, and with P_2 the price charged by firm 2 for its bundle. Given that firms 1 and 3 are in a symmetric position, we focus on NE of γ^2 such that $p_1 = p_3$.

In γ^2, the available systems are $S_{11}, S_{22}, S_{13}, S_{31}, S_{33}$, and each consumer buys the system that is cheapest for him, given p_1, P_2, p_3 and given his locations. Then, for a consumer located at (x_A, x_B), (3) reveals that the cost of S_{22} is, for each $(x_A, x_B) \in [0, 1) \times [0, 1)$,

$$C_{22}(x_A, x_B) = (x_A - \frac{1}{2})^2 + (x_B - \frac{1}{2})^2 + P_2 \tag{4}$$

and the cost of S_{ij}, for $i = 1, 3$ and $j = 1, 3$, is

$$C_{ij}(x_A, x_B) = C_i(x_A) + C_j(x_B), \qquad \text{in which}$$
$$C_1(x) = \begin{cases} (x - \frac{1}{6})^2 + p_1 & \text{if } 0 \le x < \frac{2}{3} \\ (1 - x + \frac{1}{6})^2 + p_1 & \text{if } \frac{2}{3} \le x < 1 \end{cases} \quad \text{and} \quad C_3(x) = \begin{cases} (\frac{1}{6} + x)^2 + p_3 & \text{if } 0 \le x < \frac{1}{3} \\ (x - \frac{5}{6})^2 + p_3 & \text{if } \frac{1}{3} \le x < 1 \end{cases} \tag{5}$$

Now we illustrate how the demand function is derived for each firm. To this purpose, we exploit the fact that since each consumer is characterized by two locations $x_A \in [0, 1)$ and $x_B \in [0, 1)$, the set of consumers can be considered as the square $[0, 1) \times [0, 1)$, in which locations are uniformly distributed.

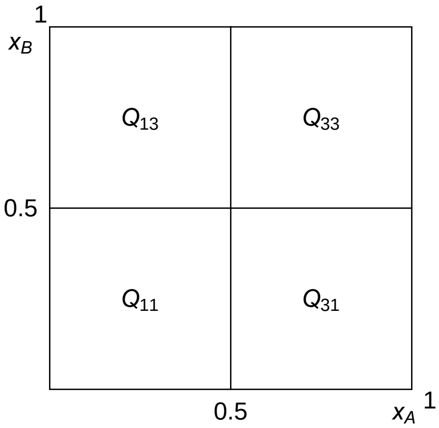
Demand function for the bundle of firm 2 In order to derive the demand function for the bundle of firm 2, we identify the set of consumers for which S_{22} is the cheapest system, that is such that $C_{22}(x_A, x_B) < \min\{C_{11}(x_A, x_B), C_{13}(x_A, x_B), C_{31}(x_A, x_B), C_{33}(x_A, x_B)\}$. We employ two steps as follows.

Step 1 We pretend that consumers can buy only from firms 1 and 3, as if there were no firm 2, and derive the resulting distribution of consumers among $S_{11}, S_{13}, S_{31}, S_{33}$. Since $p_1 = p_3$ in equilibrium and firm 1 (3) is located at $\frac{1}{6}$ (at $\frac{5}{6}$), in each market a consumer buys from firm 1 (firm 3) if he is located between 0 and $\frac{1}{2}$ (between $\frac{1}{2}$ and 1). We let Q_{ij} denote the region of consumers that buy system S_{ij}, for $i = 1, 3$ and $j = 1, 3$, when there is no firm 2. Hence

$$Q_{11} = [0, \frac{1}{2}) \times [0, \frac{1}{2}), \quad Q_{13} = [0, \frac{1}{2}) \times [\frac{1}{2}, 1), \quad Q_{31} = [\frac{1}{2}, 1) \times [0, \frac{1}{2}), \quad Q_{33} = [\frac{1}{2}, 1) \times [\frac{1}{2}, 1) \tag{6}$$

[14] The reason is that $d(x, \frac{1}{2})$ is equal to $(x - \frac{1}{2})^2$ for each $x \in [0, 1)$, whereas if $y \neq \frac{1}{2}$ then $d(x, y)$ is a piecewise defined function of x as in (2). Thus, $x_A^2 = x_B^2 = \frac{1}{2}$ simplifies $d(x_A, x_A^2) + d(x_B, x_B^2)$.

Figure 4
Consumers' purchases when S_{22} is not available and $p_1 = p_3$:



Step 2 For each region in (6) we identify the consumers that prefer S_{22} to the best alternative offered by firms 1, 3. Precisely, for $i = 1, 3$ and $j = 1, 3$ we solve $C_{22}(x_A, x_B) < C_{ij}(x_A, x_B)$ for (x_A, x_B) in Q_{ij} to determine the consumers that prefer S_{22} to S_{ij}. For instance, consider $i = 1$, $j = 3$ and let p be the common equilibrium value of p_1 and p_3. Then (5)-(4) yield $C_{13}(x_A, x_B) = (x_A - \frac{1}{6})^2 + (x_B - \frac{5}{6})^2 + 2p$ for $(x_A, x_B) \in Q_{13}$ and $C_{22}(x_A, x_B) < C_{13}(x_A, x_B)$ reduces to $x_B < \frac{1}{3} - \frac{3}{2}(P_2 - 2p) + x_A$. More generally,

$$\begin{array}{lll} \text{for } (x_A, x_B) \in Q_{11}, \quad C_{22}(x_A, x_B) < C_{11}(x_A, x_B) & \text{is equivalent to} & x_B > \frac{2}{3} + \frac{3}{2}(P_2 - 2p) - x_A \\ \text{for } (x_A, x_B) \in Q_{13}, \quad C_{22}(x_A, x_B) < C_{13}(x_A, x_B) & \text{is equivalent to} & x_B < \frac{1}{3} - \frac{3}{2}(P_2 - 2p) + x_A \\ \text{for } (x_A, x_B) \in Q_{31}, \quad C_{22}(x_A, x_B) < C_{31}(x_A, x_B) & \text{is equivalent to} & x_B > -\frac{1}{3} + \frac{3}{2}(P_2 - 2p) + x_A \\ \text{for } (x_A, x_B) \in Q_{33}, \quad C_{22}(x_A, x_B) < C_{33}(x_A, x_B) & \text{is equivalent to} & x_B < \frac{4}{3} - \frac{3}{2}(P_2 - 2p) - x_A \end{array} \tag{7}$$

We let R_{22} denote the resulting subset of $[0, 1) \times [0, 1)$, which depends on $P_2 - 2p$ as follows: R_{22} is the whole $[0, 1) \times [0, 1)$ if $P_2 - 2p < -\frac{4}{9}$, is empty if $P_2 - 2p \geq \frac{2}{9}$. If $P_2 - 2p$ is between $-\frac{4}{9}$ and $\frac{2}{9}$, then R_{22} is a more complicated convex polygon which we describe by listing its vertices. In particular, given $\mathbf{x} = (x_A, x_B) \in [0, 1) \times [0, 1)$, we use $\bar{\mathbf{x}}$ to denote the point that is obtained by permuting the coordinates of $\mathbf{x}$, that is $\bar{\mathbf{x}} = (x_B, x_A)$. It turns out that R_{22} is the octagon in Figure 5 if $-\frac{4}{9} \leq P_2 - 2p < -\frac{1}{9}$,[15] is the square in Figure 6 if $-\frac{1}{9} \leq P_2 - 2p < \frac{2}{9}$,[16] :

The set R_{22} of consumers that buy the bundle of firm 2 in γ^2 given $-\frac{4}{9} \leq P_2 - 2p < -\frac{1}{9}$ (Figure 5), or given $-\frac{1}{9} \leq P_2 - 2p < \frac{2}{9}$ (Figure 6) :

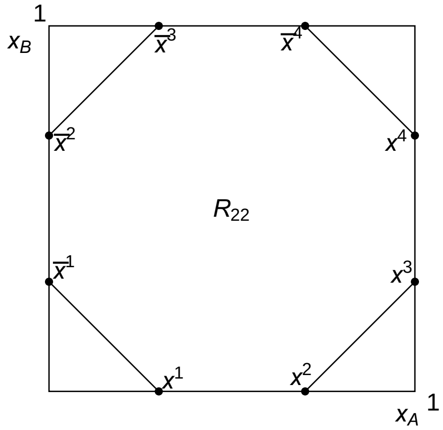


Figure 5

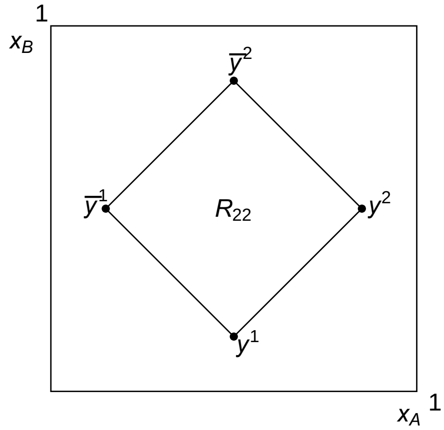


Figure 6

[15] In Figure 5, $\mathbf{x}^1 = (\lambda, 0)$, $\mathbf{x}^2 = (1 - \lambda, 0)$, $\mathbf{x}^3 = (1, \lambda)$, $\mathbf{x}^4 = (1, 1 - \lambda)$, with $\lambda = \frac{2}{3} + \frac{3}{2}(P_2 - 2p)$.

[16] In Figure 6, $\mathbf{y}^1 = (\frac{1}{2}, \lambda - \frac{1}{2})$, $\mathbf{y}^2 = (\frac{3}{2} - \lambda, \frac{1}{2})$, with $\lambda = \frac{2}{3} + \frac{3}{2}(P_2 - 2p)$.

The demand for the bundle of firm 2 is just the area of R_{22}, hence

$$D_2(P_2) = \begin{cases} 1 & \text{if } P_2 < 2p - \frac{4}{9} \\ 1 - 2(\frac{2}{3} - 3p + \frac{3}{2}P_2)^2 & \text{if } 2p - \frac{4}{9} \leq P_2 < 2p - \frac{1}{9} \\ 2(\frac{1}{3} + 3p - \frac{3}{2}P_2)^2 & \text{if } 2p - \frac{1}{9} \leq P_2 < 2p + \frac{2}{9} \\ 0 & \text{if } 2p + \frac{2}{9} \leq P_2 \end{cases} \tag{8}$$

The profit of firm 2 is $P_2 D_2(P_2)$, and we denote with $br_2(p, \gamma^2)$ the profit maximizing value of P_2, that is the best reply of firm 2 given p:[17]

$$br_2(p, \gamma^2) = \begin{cases} \frac{2}{3}p + \frac{2}{27} & \text{if } p \leq \frac{5}{36} \\ \frac{4}{3}p - \frac{8}{27} + \frac{1}{27}\sqrt{324p^2 - 144p + 70} & \text{if } p > \frac{5}{36} \end{cases} \tag{9}$$

Demand function for the products of firm 3 Here we are concerned with the demand function for the products of firm 3, which depends on p_1, P_2, p_3. Precisely, we derive the demand for firm 3 for p_3 close to p_1, which allows to obtain a first order condition for p_3 (recall that we are searching for a NE such that $p_1 = p_3$). This can be combined with the best reply function of firm 2 in (9) to identify prices p, P and a candidate NE for γ^2 such that $p_1 = p$, $P_2 = P$, $p_3 = p$. Lemma 3 establishes that this is indeed a NE of γ^2.

In deriving the demand for firm 3 for p_3 close to p_1, we follow two steps as we have done for firm 2.

Step 1 Given p_3 slightly larger than p_1, we examine the consumers' purchases when only $S_{11}, S_{13}, S_{31}, S_{33}$ are available, as if there were no firm 2. Since $p_3 > p_1$, solving $C_3(x) \leq C_1(x)$ (see (5)) reveals that the consumers buying product A_3 (B_3) are those with x_A (x_B) in the interval $[y, z)$, with $y = \frac{1}{2} + \frac{3}{4}(p_3 - p_1) > \frac{1}{2}$ and $z = 2 - 2y < 1$; conversely, the consumers with x_A (x_B) in $[0, y) \cup [z, 1)$ buy product A_1 (B_1). As a consequence, we define the sets Q_{11}, Q_{13}, Q_{31}, Q_{33} as follows (see Figure 7):

$$\begin{aligned} Q_{11} &= ([0, y) \cup [z, 1)) \times ([0, y) \cup [z, 1)), \qquad & Q_{13} &= ([0, y) \cup [z, 1)) \times [y, z), \\ Q_{31} &= [y, z) \times ([0, y) \cup [z, 1)), \qquad & Q_{33} &= [y, z) \times [y, z) \end{aligned} \tag{10}$$

Step 2 Since we are interested in the demand for firm 3, we neglect Q_{11} but for the other regions in (10) we identify the consumers that prefer a system offered by firms 1 and 3 to S_{22}. Precisely, for $ij = 13, 31, 33$ we solve $C_{ij}(x_A, x_B) < C_{22}(x_A, x_B)$ for $(x_A, x_B) \in Q_{ij}$. For instance, (4)-(5) reveal that in Q_{33} the inequality $C_{33}(x_A, x_B) < C_{22}(x_A, x_B)$ reduces to $x_B > \frac{4}{3} + \frac{3}{2}(2p_3 - P_2) - x_A$. Therefore, the set of consumers that buy S_{33} is given by region R_{33} in Figure 8.[18] Arguing likewise for Q_{13} and Q_{31} shows that the set of consumers

[17] We need to distinguish $p \leq \frac{5}{36}$ from $p > \frac{5}{36}$ because in the first case it is optimal for firm 2 to choose P_2 that makes R_{22} equal to a square as in Figure 6, and the optimal P_2 is obtained using $D_2(P_2) = 2(\frac{1}{3} + 3p - \frac{3}{2}P_2)^2$; in the second case it is optimal for firm 2 to choose P_2 which makes R_{22} equal to an octagon as in Figure 5, and the optimal P_2 is obtained using $D_2(P_2) = 1 - 2(\frac{2}{3} - 3p + \frac{3}{2}P_2)^2$.

[18] In Figure 8, $\mathbf{x}^1 = (y, 3y - \eta)$, $\mathbf{x}^2 = (\eta - y, y)$ with $\eta = \frac{4}{3} + \frac{3}{2}(2p_3 - P_2)$.

that buy just one product from firm 3 is $R_{13} \cup R_{31}$ in Figure 8.[19]

The sets $Q_{11}, Q_{13}, Q_{31}, Q_{33}$ in (10) (Figure 7) and the sets R_{13}, R_{31}, R_{33} (Figure 8)

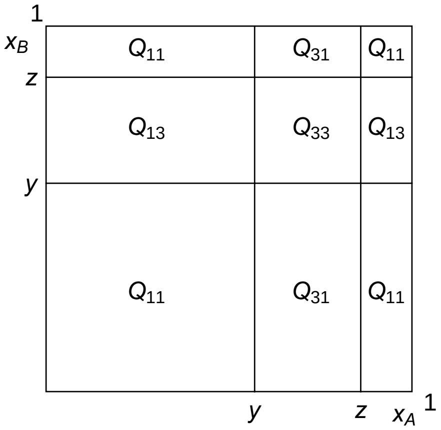


Figure 7

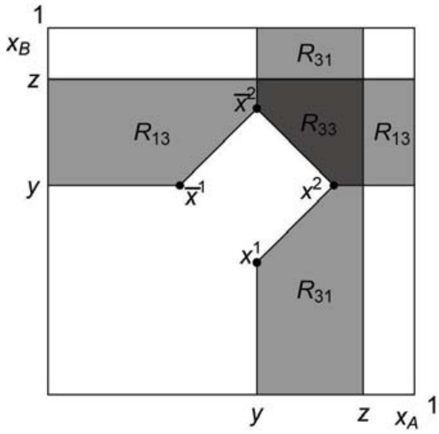


Figure 8

Thus, the demand for firm 3 given p_3 slightly larger than p_1 is equal to twice the area of R_{33} plus the area of $R_{13} \cup R_{31}$. This yields

$$D_3(p_3) = 1 + \frac{9}{2}(p_1 - p_3) - \frac{9}{2}(\frac{2}{9} + p_1 + p_3 - P_2)^2 \tag{11}$$

In fact, it turns out that the demand for firm 3 has the expression in (11) also for p_3 slightly smaller than p_1. Therefore, for p_3 close to p_1 the profit function of firm 3 is $p_3 D_3(p_3)$ with $D_3(p_3)$ in (11). From this we can derive the first order condition for p_3, which must be satisfied at $p_3 = p_1$. Combining it with (9) identifies the equilibrium prices.

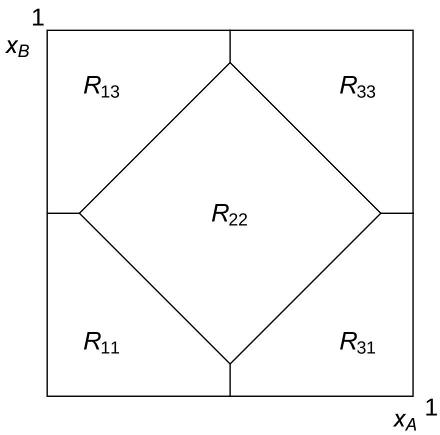
Lemma 3 *When $n = 3$, consider the subgame γ^2 which is entered if firm 2 chooses PB and firms 1 and 3 choose IP at stage one. In γ^2 there exists a NE such that*

$$p_1^* = p_3^* = \frac{3\sqrt{4681} - 137}{720} = 0.0948, \qquad P_2^* = \frac{\sqrt{4681} - 19}{360} = 0.1373 \tag{12}$$

The equilibrium profit of firm 2 is 0.0466*; the equilibrium profit of each other firm is* 0.0626*.*

Figure 9 represents the equilibrium distribution of consumers among the available systems:[20]

Figure 9
The distribution of consumers in the NE of γ^2 among $S_{11}, S_{13}, S_{31}, S_{33}, S_{22}$



[19] Figure 8 is obtained under the assumption that $P_2 - 2p_3 > -\frac{1}{9}$, which indeed holds in equilibrium (see Lemma 3 below), otherwise the vertical coordinate of $\bar{\mathbf{x}}^2$ (the horizontal coordinate of $\mathbf{x}^2$) would be greater than z and R_{33} would be a triangle.

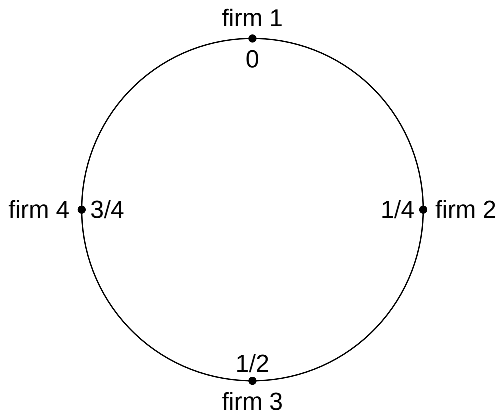
[20] The vertices in Figure 9 are the vertices in Figure 6 (see footnote 16) with the equilibrium prices in Lemma 3.

From Lemmas 1-3 we see that in γ^2 the profit of each firm is smaller than in $\gamma^{\varnothing}$. Moreover, firm 2 (firm 1, firm 3) has a lower (higher) profit in γ^2 than in γ^{123}. In Section 4 we examine these results in some detail and we discuss their consequences on firms' choices at stage one.

3.3.2 Asymmetric subgames when $n = 4$

When there are four firms, to fix the ideas we assume that firms are located as follows on both circles, that is firm i is located at $x = i - \frac{1}{4}$ for $i = 1, ..., 4$:

Figure 10
Distribution of firms over the circle for each product when $n = 4$:



When $n = 4$, the asymmetric subgames of Γ are the following:

- (a) $\gamma^1, \gamma^2, \gamma^3, \gamma^4$, that is the subgames which are entered if only one firm has chosen PB;
- (b) $\gamma^{12}, \gamma^{23}, \gamma^{34}, \gamma^{41}$, that is the subgames which are entered if two adjacent firms have chosen PB, and the other two firms have chosen IP;
- (c) γ^{13}, γ^{24}, that is the subgames which are entered if two non-adjacent firms have chosen PB, and the other two firms have chosen IP

As we have mentioned in Subsection 3.3, in these subgames firms are asymmetric not only because of different choices at stage one, but (in some cases) because of their locations. Precisely, in subgames $\gamma^1, \gamma^2, \gamma^3, \gamma^4$ the three firms that have chosen IP are not in a symmetric position because two of them are adjacent to a firm that bundles and the third is adjacent to firms that do not bundle. Thus, only two firms' individual products face direct competition from a bundle. Moreover, we need to distinguish $\gamma^{12}, \gamma^{23}, \gamma^{34}, \gamma^{41}$ from γ^{13}, γ^{24} because if exactly two firms choose PB at stage one, then it is important whether the two firms are adjacent or not.

The analysis for the asymmetric subgames when $n = 4$ is longer and more complicated than for the subgame γ^2 we examined in Subsection 3.3, and it is available from the authors upon request. Next lemma identifies the equilibrium prices and profits for just one subgame for each of the classes (a), (b), (c) listed above, as analogous results hold for the other subgames in the same class.

Lemma 4 *Consider game* Γ *when* $n = 4$
(i) in subgame γ^3 *there exists a NE such that* $p_1 = 0.0621$, $p_2 = 0.0616$, $P_3 = 0.0827$, $p_4 = 0.0616$. *The equilibrium profits are* $\pi_1 = 0.0308$, $\pi_2 = 0.0328$, $\Pi_3 = 0.0181$, $\pi_4 = 0.0328$.
(ii) in subgame γ^{23} *there exists a NE such that* $p_1 = 0.06$, $P_2 = P_3 = 0.1012$, $p_4 = 0.06$. *The equilibrium profits are* $\pi_1 = 0.0313$, $\Pi_2 = \Pi_3 = 0.0242$, $\pi_4 = 0.0313$.
(iii) in subgame γ^{13} *there exists a NE such that* $P_1 = 0.0824$, $p_2 = 0.061$, $P_3 = 0.0824$, $p_4 = 0.061$. *The equilibrium profits are* $\Pi_1 = 0.0179$, $\pi_2 = 0.0345$, $\Pi_3 = 0.0179$, $\pi_4 = 0.0345$.

4 The marketing stage

In this section we examine the marketing stage in which each firm chooses its pricing regime, either IP or PB. To this purpose, we study the reduced game with simultaneous moves in which each firm's set of feasible actions is {IP, PB} and given any action profile $(a_1, a_2, ..., a_n) \in \{\text{IP,PB}\}^n$, the firms' profits are given by the equilibrium profits in the price competition stage which is entered given $(a_1, a_2, ..., a_n)$. Note that (PB, PB, ..., PB) is a NE of the reduced game for each n. Indeed, if all firms different from firm i play PB, then firm i has no incentive to deviate by choosing IP as $\gamma^{N\setminus\{i\}}$ is equivalent to γ^N. Hence, there always exists a SPNE of Γ in which each firm bundles; we call it the *trivial* SPNE.

4.1 The reduced game when $n = 3$ and $n = 4$

Using Lemmas 1-3 we obtain the following reduced game when $n = 3$, in which firm 1 chooses a row, firm 2 chooses a column, firm 3 chooses a matrix:[21]

$a_3 = \text{IP}$

	$a_2 = \text{IP}$	$a_2 = \text{PB}$
$a_1 = \text{IP}$	0.0741,0.0741,0.0741	0.0626,0.0466,0.0626
$a_1 = \text{PB}$	0.0466,0.0626,0.0626	0.0556,0.0556,0.0556

$a_3 = \text{PB}$

	$a_2 = \text{IP}$	$a_2 = \text{PB}$
$a_1 = \text{IP}$	0.0626,0.0626,0.0466	0.0556,0.0556,0.0556
$a_1 = \text{PB}$	0.0556,0.0556,0.0556	0.0556,0.0556,0.0556

It is immediate to see that in this game, for each firm IP weakly dominates PB because $0.0741 > 0.0466$, $0.0626 > 0.0556$.

Likewise, when $n = 4$, Lemmas 1, 2, 4 allow to determine each firm's profit as a function of $(a_1, ..., a_4)$. However, instead of providing the complete normal form of the reduced game, we represent only the profit of a firm – firm 3, without loss of generality – as a function of $a_1, ..., a_4$. In particular, in the table below the second and third row represent each one choice of firm 3 (a_3 =IP or a_3 =PB) and each column – except the first – represents one of the possible triples a_1, a_2, a_4:

3\1,2,4	IP,IP,IP	PB,IP,IP	IP,PB,IP	IP,IP,PB	PB,PB,IP	PB,IP,PB	IP,PB,PB	PB,PB,PB
$a_3 = \text{IP}$	0.0313	0.0308	0.0328	0.0328	0.0313	0.0313	0.0345	0.025
$a_3 = \text{PB}$	0.0181	0.0179	0.0242	0.0242	0.025	0.025	0.025	0.025

Again, IP weakly dominates PB for firm 3. Hence, the same result holds for each other firm as well.

Proposition 1 *In the reduced game for Γ when $n = 3$ or $n = 4$, IP weakly dominates PB for each firm and the unique non-trivial SPNE of Γ is such that each firm plays IP at stage one.*[22]

Proposition 1 establishes that unless firms coordinate on the trivial SPNE, the only equilibrium outcome is that all firms choose IP and competition occurs among individual products. We have remarked in Subsection 3.2 that all firms are better off in $\gamma^{\varnothing}$ than in γ^N, hence starting from (IP, IP, ..., IP) firms have no collective incentive to move (PB, PB, ..., PB). Proposition 1 establishes that no individual incentive for a firm to bundle exists either, regardless of the marketing regimes chosen by the other firms. In the rest of this section we explore in some detail the causes of this result when $n = 3$; see footnote 24 for a remark about the case of $n = 4$.

[21] In each entry, the i^{th} number is the profit of firm i, for $i = 1, 2, 3$.

[22] The complete SPNE strategies (which include each firm's behavior at stage two) are obtained from Lemmas 1-3 when $n = 3$, from Lemmas 1, 2, 4 when $n = 4$.

4.2 The unprofitability of bundling when $n = 3$

In this subsection we explain why, when $n = 3$, PB is weakly dominated by IP for each firm. We rely on two notions described by Hurkens *et al.* (2018): the *demand size effect* and the *demand elasticity effect*.

Bundling is unprofitable when no other firm bundles Without loss of generality, we examine firm 2 and its incentive to bundle given a_1 =IP, a_3 =IP. For the demand size effect we start from the NE in $\gamma^{\varnothing}$, in which each firm offers separate products, each at price $p^{\varnothing} = \frac{1}{9}$. Now suppose firm 2 bundles its products and sets P_2 equal to $2p^{\varnothing} = \frac{2}{9}$, the total equilibrium price of A_2, B_2 in $\gamma^{\varnothing}$; firms 1, 3 keep offering separate products at the unit price $p^{\varnothing}$. The demand size effect inquires the change in profit for each firm determined by bundling of firm 2 with unchanged prices. From (8) we know that $P_2 = 2p^{\varnothing}$ and $p_1 = p_3 = p^{\varnothing}$ make the demand for the bundle of 2 equal to $\frac{2}{9}$, smaller than $\frac{1}{3}$, the demand for each product of firm 2 in the NE in $\gamma^{\varnothing}$. Therefore, firm 2 loses (firms 1, 3 gain) market share and profit with respect to $\gamma^{\varnothing}$.

In order to see why, notice that bundling makes unavailable the systems $S_{12}, S_{21}, S_{32}, S_{23}$, hence each consumer who buys one of these systems in $\gamma^{\varnothing}$ must change his purchase in γ^2,[23] and the revenue of firm 2 comes uniquely from the sale of S_{22}. Figure 11 represents the sets of the consumers that buy one or both products of firm 2 in $\gamma^{\varnothing}$, denoted with $R_{ij}^{\varnothing}$ for $ij = 12, 21, 23, 32, 22$. Figure 12 represents the set R_{22} of consumers that buy S_{22} in γ^2 given $p_1 = p_3 = p^{\varnothing}$, $P_2 = 2p^{\varnothing}$. This set includes $R_{22}^{\varnothing}$ and a subset of $R_{12}^{\varnothing}$, of $R_{21}^{\varnothing}$, of $R_{23}^{\varnothing}$, of $R_{32}^{\varnothing}$; we focus on $R_{32}^{\varnothing}$ to fix the ideas (similar arguments apply to $R_{12}^{\varnothing}, R_{21}^{\varnothing}, R_{23}^{\varnothing}$). For each consumer in $R_{32}^{\varnothing}$, bundling doubles firm 2's revenue from the consumer if he buys S_{22} (i.e., if he is in R_{22}), but reduces the revenue to 0 if the consumer buys a different system. As Figure 12 suggests, the consumers in $R_{32}^{\varnothing}$ that belong to R_{22} are fewer than those that do not; thus, relative to the set $R_{32}^{\varnothing}$, bundling makes firm 2 lose more consumers than those that end up buying S_{22}. For instance, a consumer located at $\mathbf{x} = (0.8, 0.4) \in R_{32}^{\varnothing}$ buys S_{32} under $\gamma^{\varnothing}$. But S_{32} becomes unavailable if firm 2 bundles, and for the consumer S_{31} is more convenient than S_{22} because these systems have the same monetary cost $2p^{\varnothing}$ but $\mathbf{x}$ is closer to S_{31} than to S_{22}. Therefore the demand size effect reduces firm 2's market share and profit.

The sets of consumers that buy at least one product of firm 2 in the NE of $\gamma^{\varnothing}$ (Figure 11), and the set of consumers that buy S_{22} in γ^2 given $p_1 = p_3 = p^{\varnothing}$, $P_2 = 2p^{\varnothing}$ (Figure 12)

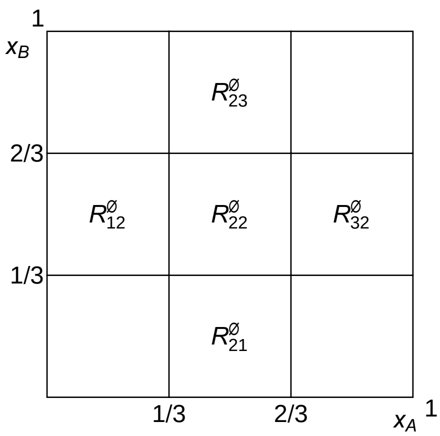


Figure 11

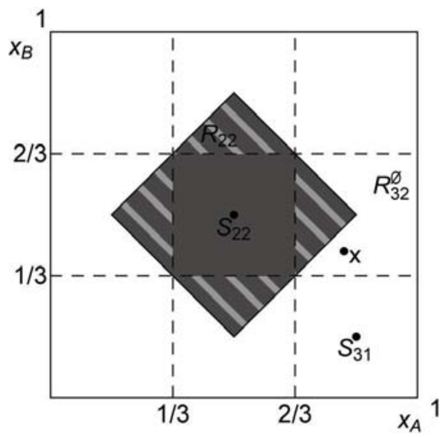


Figure 12

The above analysis neglects the demand elasticity effect, that is the firms' incentives to change prices given that firm 2 bundles. Precisely, from (9) we see that given $p_1 = p_3 = p^{\varnothing}$, the optimal price for firm 2 is $\frac{4}{27}$, smaller than $2p^{\varnothing}$; thus firm 2 wants to reduce P_2. This occurs because the lower demand of firm 2 reduces the loss from reducing the price to inframarginal consumers, but also because firm 2's demand in γ^2

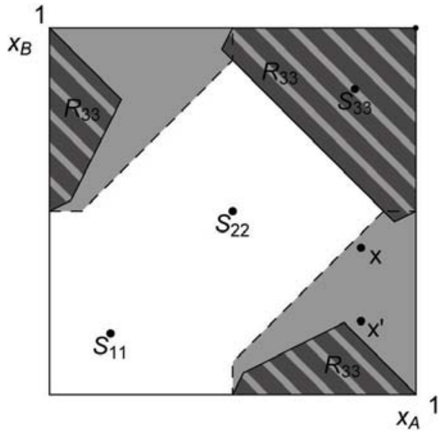
[23] Conversely, bundling by firm 2 does not change the purchase of any consumer that in $\gamma^{\varnothing}$ buys one of the other systems, as they remain available and at the same price.

is more reactive to a decrease in price than firm 2's demand in $\gamma^{\varnothing}$. Firms 1 and 3, if P_2 were fixed at $2p^{\varnothing}$, would want to increase slightly p_1, p_3 above $p^{\varnothing}$. But since prices are strategic complements, the reduction of P_2 to $\frac{4}{27}$ induces 1,3 to reduce p_1, p_3 below $p^{\varnothing}$. This pushes firm 2 to further reduce P_2, and the NE is reached at the prices in Lemma 3.

Combining the two effects yields the equilibrium outcome under γ^2, in which firm 2's market share is 0.3392. Although this is greater than $\frac{1}{3}$, the price of S_{22} is low enough that firm 2 is worse off with respect to $\gamma^{\varnothing}$, and also with respect to γ^{123}. The stronger price competition hurts also firms 1, 3 as they have about the same market share as in $\gamma^{\varnothing}$ but charge a price for each product lower than $p^{\varnothing}$.[24]

Bundling is unprofitable when only another firm bundles Now we suppose that a_1 =IP, a_2 =PB and illustrate why for firm 3 it is unprofitable to play PB. If firm 3 bundles its products, then γ^{123} is entered. We examine the demand size effect supposing that $P_1 = P_3 = 2p^*$ and $P_2 = P_2^*$ ($p^* = 0.0948$, $P_2^* = 0.1373$ as in the NE in γ^2: see Lemma 3). Figure 13 describes how the set of customers of firm 3 changes in moving from γ^2 to γ^{123} with unchanged prices; see also Figure 9. Precisely, we denote with R_{ij}^2 the set of consumers that buy S_{ij} in the NE of γ^2, for $ij = 13, 31, 33$, and represent the boundaries of these sets with dashed segments; the solid segments are the boundaries of the set R_{33} of consumers that buy S_{33} in γ^{123}.[25] Firm 3 keeps all the consumers that buy S_{33} in γ^2 but, as Figure 13 suggests, loses most of the consumers in $R_{13}^2 \cup R_{31}^2$ because they buy S_{22} or S_{11} rather than S_{33}.[26] For example, the consumer located at $\mathbf{x} = (0.85, 0.4)$ and the consumer located at $\mathbf{x}' = (0.85, 0.2)$ both buy S_{31} in γ^2, but in γ^{123} the first consumer buys S_{22}, the second consumer buys S_{11}. Hence, the demand size effect is negative for firm 3.

Figure 13
The set R_{33} of consumers that buy S_{33} in γ^{123}, given $P_1 = 2p^*$, $P_2 = P_2^*$, $P_3 = 2p^*$ (p^*, P_2^* are the NE prices in γ^2: see Lemma 3)



Also in this case there is a demand elasticity effect that modifies the firms' incentives about pricing. In particular, at $P_2 = P_2^*$, $P_1 = P_3 = 2p^*$ firms 1 and 3 want to reduce P_1, P_3, whereas firm 2 wants to increase P_2.[27] Consistently with these incentives, the equilibrium price for each bundle in γ^{123} is $\frac{1}{6}$, such that $P_2^* < \frac{1}{6} < 2p^*$. At the equilibrium prices, the market share of firm 3 is slightly higher than in γ^2, $\frac{1}{3}$ instead of 0.3304, but the price $\frac{1}{6}$ of its bundle is smaller than $2p^*$; this makes its profit 0.0556 smaller than the profit 0.0626 under γ^2. Thus, firm 3 has no incentive to bundle when a_1 =IP, a_2 =PB.

[24] A similar argument applies to the subgame γ^3 when $n = 4$. Firm 3 suffers a sizable negative demand size effect (which benefits firms 2, 4 but does not affect firm 1), and the demand elasticity effect induces firm 3 to decrease P_3 significantly. This induces firms 2, 4 to reduce p_2, p_4, which further hurts firm 3. In the NE of γ^3 (see Lemma 4(i)), firms 2, 4 are better off than in $\gamma^{\varnothing}$ because of the market share lost by firm 3; firm 1 is worse off because of the more aggressive pricing by firms 2, 4.

[25] In Subsection 5.4 and in the proof of Lemma 8 we provide details about the derivation of R_{33}.

[26] Precisely, the demand for the bundle of firm 3 is 0.2779, whereas 3's market share in γ^2 is 0.3304 in both markets.

[27] This occurs because in γ^{123}, firm 2 faces softer competition than in γ^2 as systems S_{13}, S_{31} are not available. In Subsection 5.5 we provide more details about this effect.

5 An asymmetric setting

In this section we examine a setting in which one firm is dominant with respect to the other firms in the sense that it offers products with higher quality than its competitors' products. Our purpose is to inquire whether this sort of asymmetry among firms generates incentives to bundle, or instead leads to results analogous to Proposition 1.

For simplicity we suppose that there are three firms, and (without loss of generality) that (i) firm 2 is the dominant firm; (ii) firms are located, in both circumferences, as described by Figure 3. The dominance of firm 2 is represented by a higher gross utility, $V+\alpha$ instead of V, with $\alpha>0$, that each consumer receives if he consumes a product of firm 2 rather than a product of another firm. Equivalently, we can think that the dominance of firm 2 reduces by α the cost of S_{ij} if this system includes exactly one product of firm 2, reduces by 2α the cost of S_{22}. For instance, if firm 2 chooses PB, then the cost of S_{22} for a consumer located at (x_A,x_B) is $d(x_A,\frac{1}{2})+d(x_B,\frac{1}{2})+P_2-2\alpha$. We use Γ_α to denote the game which differs from Γ only because firm 2 is dominant, and with $\gamma_\alpha^{N'}$ the stage two subgame of Γ_α that is entered if $N'\subseteq\{1,2,3\}$ is the set of the firms that at stage one choose PB.

We remark that the dominance of firm 2 introduces an ex ante asymmetry among firms which was absent in the previous sections (but of course, $\Gamma_0=\Gamma$). In particular, Lemmas 1 and 2 do not apply to $\gamma_\alpha^{\varnothing}$ and to γ_α^{123}, respectively, and we need to distinguish between γ_α^{2} and γ_α^{3} (γ_α^{1} is equivalent to γ_α^{3} up to a relabeling of firms) because it makes a difference if the unique bundling firm is the dominant firm or a dominated firm. However, it is still the case that $\gamma_\alpha^{12},\gamma_\alpha^{13},\gamma_\alpha^{23}$ are each equivalent to γ_α^{123}.

5.1 Independent sales: Subgame $\gamma_\alpha^{\varnothing}$

Here we consider the subgame that is played if each firm has chosen IP at stage one. Under independent pricing, competition for product A is independent of competition for product B even though firm 2 is dominant. Hence, we focus on the market for product A and for a consumer located at x_A the cost of product A_j, for $j=1,3$, is $C_j(x_A)$ in (5); the cost of product A_2 is $C_2(x_A)=(x_A-\frac{1}{2})^2+p_2-\alpha$.

Since firms 1, 3 are in a symmetric position, we examine NE of $\gamma_\alpha^{\varnothing}$ such that they charge the same price. Given $p_1=p_3=p$, in order to derive the demand function for firm 2 we solve the inequality $C_2(x_A)<\min\{C_1(x_A),C_3(x_A)\}$, in which $\min\{C_1(x_A),C_3(x_A)\}=C_1(x_A)$ if $x_A\in[0,\frac{1}{2})$, $\min\{C_1(x_A),C_3(x_A)\}=C_3(x_A)$ if $x_A\in[\frac{1}{2},1)$. Hence, $C_2(x_A)<\min\{C_1(x_A),C_3(x_A)\}$ holds for each $x_A\in[0,1)$ if $p_2<p+\alpha-\frac{2}{9}$, holds for $x_A\in(\frac{1}{3}+\frac{3}{2}(p_2-p-\alpha),\frac{2}{3}+\frac{3}{2}(p+\alpha-p_2))$ if $p+\alpha-\frac{2}{9}\le p_2<p+\alpha+\frac{1}{9}$, is violated for each $x_A\in[0,1)$ if $p_2\ge p+\alpha+\frac{1}{9}$. Therefore the demand function for firm 2 is

$$D_2(p_2)=\begin{cases}1 & \text{if } p_2<p+\alpha-\frac{2}{9}\\ 3p+3\alpha-3p_2+\frac{1}{3} & \text{if } p+\alpha-\frac{2}{9}\le p_2<p+\alpha+\frac{1}{9}\\ 0 & \text{if } p+\alpha+\frac{1}{9}\le p_2\end{cases}\tag{13}$$

and it is immediate to see that the best reply for firm 2 is

$$br_2(p,\gamma_\alpha^{\varnothing})=\begin{cases}\frac{1}{18}+\frac{1}{2}p+\frac{1}{2}\alpha & \text{if } p+\alpha<\frac{5}{9}\\ p+\alpha-\frac{2}{9} & \text{if } p+\alpha\ge\frac{5}{9}\end{cases}\tag{14}$$

Now we derive the demand function of firm 3. Assume that firms 1 and 3 have both a positive market share in NE. Given the price p charged by firm 1 in equilibrium, it turns out that for p_3 close to p_1,

$C_3(x_A) < \min\{C_1(x_A), C_2(x_A)\}$ reduces to $x_A \in (\frac{2}{3} + \frac{3}{2}(p_3 + \alpha - p_2), 1 - \frac{3}{2}(p_3 - p))$;[28] hence

$$D_3(p_3) = \frac{1}{3} + \frac{3}{2}(p + p_2 - \alpha - 2p_3) \tag{15}$$

From (15) we derive a first order condition for p_3, which combined with (14) delivers the equilibrium prices when all firms have a positive market share. Next lemma also determines when firm 2 captures the whole market.

Lemma 5 *In game Γ_α, consider the subgame $\gamma_\alpha^\varnothing$ which is entered if each firm chooses IP at stage one. In this subgame, for each market there exists a NE such that $p_1 = p_3 = p^*$ and $p_2 = p_2^*$ with*

$$p^* = \frac{1}{9} - \frac{1}{5}\alpha, \qquad p_2^* = \frac{2}{5}\alpha + \frac{1}{9} \qquad \textit{if } \alpha \in (0, \frac{5}{9}) \tag{16}$$

$$p^* = 0, \qquad p_2^* = \alpha - \frac{2}{9} \qquad \textit{if } \alpha \geq \frac{5}{9} \tag{17}$$

In the following of this section we only consider the case in which each firm has positive market share and profit in $\gamma_\alpha^\varnothing$, that is we restrict to $\alpha \in (0, \frac{5}{9})$.

5.2 Bundling by firm 2 only: Subgame γ_α^2

In subgame γ_α^2, only the dominant firm bundles. Therefore, this game is similar to the subgame γ^2 we examined in Subsection 3.3, with the only difference that the bundling firm offers products with higher quality than its competitors' products. This has the effect that $C_{22}(x_A, x_B)$ is not given by (4), but is equal to $(x_A - \frac{1}{2})^2 + (x_B - \frac{1}{2})^2 + P_2 - 2\alpha$.

As in Subsection 3.3, we consider NE in which firm 1 and firm 3 charge a same price for each single product they offer. If p is the common value of p_1 and p_3, it is simple to obtain the demand function and the best reply of firm 2 because firm 2's per-product advantage α has the same effect as an increase in p by α. As a consequence, from (8) and (9) we obtain

$$D_2(P_2) = \begin{cases} 1 & \text{if } P_2 < 2p + 2\alpha - \frac{4}{9} \\ 1 - 2(\frac{2}{3} - 3(p+\alpha) + \frac{3}{2}P_2)^2 & \text{if } 2p + 2\alpha - \frac{4}{9} \leq P_2 < 2p + 2\alpha - \frac{1}{9} \\ 2(\frac{1}{3} + 3(p+\alpha) - \frac{3}{2}P_2)^2 & \text{if } 2p + 2\alpha - \frac{1}{9} \leq P_2 < 2p + 2\alpha + \frac{2}{9} \\ 0 & \text{if } 2p + 2\alpha + \frac{2}{9} \leq P_2 \end{cases} \tag{18}$$

and

$$br_2(p, \gamma_\alpha^2) = br_2(p+\alpha, \gamma^2) = \begin{cases} \frac{2}{3}p + \frac{2}{3}\alpha + \frac{2}{27} & \text{if } p + \alpha \leq \frac{5}{36} \\ \frac{4}{3}(p+\alpha) - \frac{8}{27} + \frac{1}{27}\sqrt{324(p+\alpha)^2 - 144(p+\alpha) + 70} & \text{if } p + \alpha > \frac{5}{36} \end{cases} \tag{19}$$

For firm 3 we can argue like in Subsection 3.3 to derive the demand function for p_3 slightly larger than p_1. Precisely, we pretend momentarily that S_{22} is unavailable and find that Q_{ij} in (10) is the set of consumers that buy S_{ij}, for $ij = 11, 13, 31, 33$, with $y = \frac{1}{2} + \frac{3}{4}(p_3 - p_1)$, $z = 2 - 2y$. Then for $ij = 13, 31, 33$ we solve $C_{ij}(x_A, x_B) < C_{22}(x_A, x_B)$ for $(x_A, x_B) \in Q_{ij}$ to identify the set of consumers that prefer S_{ij} to S_{22}. When α is about 0, the set R_{33} of consumers that buy S_{33} and the set $R_{13} \cup R_{31}$ of consumers that buy just one product from firm 3 are similar to the analogous sets in Figure 8.[29] As a consequence,

$$D_3(p_3) = 1 + \frac{9}{2}(p_1 - p_3) - \frac{9}{2}(\frac{2}{9} + 2\alpha + p_1 + p_3 - P_2)^2 \tag{20}$$

[28] If p_3 is slightly larger than p, then $\frac{2}{3} + \frac{3}{2}(p_3 + \alpha - p_2) < 1 - \frac{3}{2}(p_3 - p)$ because firm 3 has a positive market share in equilibrium, hence $p + \alpha - \frac{2}{9} < p_2$.

[29] But in the vertices described in footnote 18, η is equal to $\frac{4}{3} + \frac{3}{2}(2p_3 + 2\alpha - P_2)$.

The profit of firm 3 is $p_3 D_3(p_3)$, hence we derive a first order condition for p_3, which combined with (19) (when $p+\alpha \le \frac{5}{36}$) identifies the equilibrium prices when α is close to zero: see p^*, P_2^* in (22) below in Lemma 6. In this case the set $[0,1) \times [0,1)$ of consumers is partitioned among the available systems as described by Figure 14, which is similar to Figure 9:[30]

Equilibrium partition of consumers in γ_α^2 among $S_{11}, S_{22}, S_{13}, S_{31}, S_{33}$ when $\alpha \le \frac{13}{180}$ (Figure 14) and when $\alpha \in (\frac{13}{180}, \frac{5}{9})$ (Figure 15) :

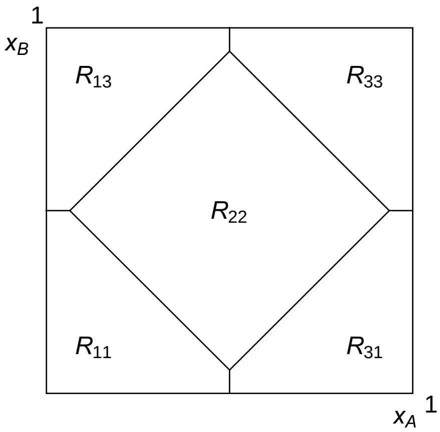


Figure 14

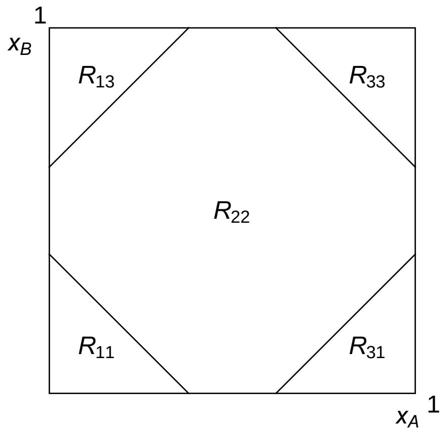
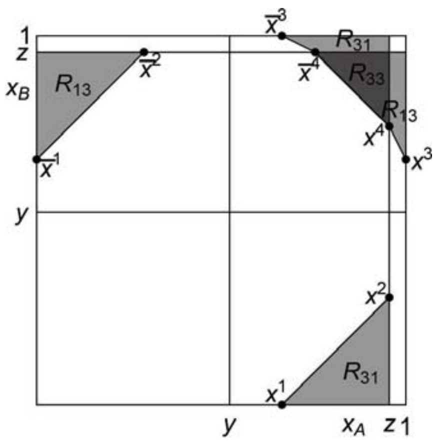


Figure 15

In order for the equilibrium prices to be given by (22), it is necessary that the set R_{22} is a square as in Figure 14. This is the case only if $P_2^* \ge 2p^* + 2\alpha - \frac{1}{9}$, but p^*, P_2^* in (22) violate this condition when $\alpha > \frac{13}{180}$. In such case, the consumers partition in equilibrium as described by Figure 15.[31] In this case, when p_3 is slightly larger than p_1 the sets R_{33} and $R_{13} \cup R_{31}$ are described in Figure 16.[32]

Figure 16
The sets R_{33}, R_{13}, R_{31} when $\alpha > \frac{13}{180}$ and p_3 is slightly larger than p_1 :



The demand for firm 3, $D_3(p_3)$, is the area of $R_{13} \cup R_{31}$ plus twice the area of R_{33}, hence

$$D_3(p_3) = \frac{63}{2}p_3^2 + (45\alpha - 18p_1 - \frac{45}{2}P - 10)p_3 + \frac{9}{2}P^2 + \frac{9}{2}Pp_1 - 18P\alpha + 4P + \frac{9}{2}p_1^2 - 9p_1\alpha + 18\alpha^2 + 2p_1 - 8\alpha + \frac{8}{9} \quad (21)$$

From this we derive a first order condition for p_3 and combine it with (19) (when $p+\alpha > \frac{5}{36}$) to obtain the equilibrium prices (23) in next lemma.

[30] The vertices in Figure $\gamma^2_{\alpha(1)}$ are the vertices in footnote 16 with $\lambda = \frac{2}{3} + \frac{3}{2}(P_2^* - 2\alpha - 2p^*)$ and p^*, P_2^* given by (22).

[31] The vertices of the octagon R_{22} are the vertices in footnote 15 with $\lambda = \frac{2}{3} + \frac{3}{2}(P_2^* - 2\alpha - 2p^*)$ and p^*, P_2^* in (23).

[32] In Figure 16, $\mathbf{x}^1 = (\lambda, 0)$, $\mathbf{x}^2 = (z, z-\lambda)$, $\mathbf{x}^3 = (1, \lambda)$, $\mathbf{x}^4 = (z, \eta - z)$, with $\lambda = \frac{1}{3} + \frac{3}{2}(p + p_3 + 2\alpha - P_2)$.

Lemma 6 *In game Γ_α, consider the subgame γ_α^2 which is entered if firm 2 chooses PB and firms 1 and 3 choose IP at stage one. In γ_α^2 there exists a NE with $p_1 = p_3 = p^*$ and $P_2 = P_2^*$ such that*

$$\begin{aligned} p^* &= \tfrac{1}{240}\sqrt{5184\alpha^2 + 10\,224\alpha + 4681} - \tfrac{7}{10}\alpha - \tfrac{137}{720} \\ P_2^* &= \tfrac{1}{360}\sqrt{5184\alpha^2 + 10\,224\alpha + 4681} + \tfrac{1}{5}\alpha - \tfrac{19}{360} \end{aligned} \qquad \text{if } \alpha \le \tfrac{13}{180} \tag{22}$$

$$\begin{aligned} p^* &= \tfrac{1}{855}\sqrt{8100\alpha^2 - 3600\alpha + 2110} - \tfrac{2}{19}\alpha + \tfrac{4}{171} \\ P_2^* &= \tfrac{24}{19}\alpha + \tfrac{7}{855}\sqrt{8100\alpha^2 - 3600\alpha + 2110} - \tfrac{16}{57} \end{aligned} \qquad \text{if } \tfrac{13}{180} < \alpha < \tfrac{5}{9} \tag{23}$$

5.3 Bundling by firm 3 only: Subgame γ_α^3

Here we study the case in which only firm 1 or only firm 3 bundles. Since the subgame γ_α^1 is equivalent to γ_α^3 (up to a relabelling of firms), we examine γ_α^3. This subgame is more complicated than γ_α^2 as there is no symmetry between any two firms: firm 2 has a quality advantage firms 1, 3 do not have, and firm 3 bundles but the others do not. In γ_α^3, a NE is a triplet (p_1^*, p_2^*, P_3^*).

If α is close to 0, then γ_α^3 is only slightly different from the subgame γ^2 examined in Subsection 3.3 (apart from the fact that in γ_α^3 it is firm 3 that bundles rather than firm 2). This suggests that given p^* and P_2^* in Lemma 3, for α close to 0 the equilibrium prices p_1^*, p_2^* are close to p^*, and P_3^* is close to P_2^*. This is useful because it is cumbersome to derive the complete demand functions in γ_α^3. It is simpler to derive them for p_1, p_2 close to p^* and P_3 close to P_2^*. Precisely, in this case we obtain

$$\begin{cases} D_1(p_1) = \begin{array}{c} (9P_3 - 9p_2 + 9\alpha - \frac{13}{2})p_1 - \frac{9}{2}p_1^2 - \frac{9}{2}\alpha^2 - 9\alpha P_3 + \frac{7}{9} \\ +9\alpha p_2 - \frac{5}{2}\alpha - \frac{9}{2}P_3^2 + 9P_3 p_2 + 2P_3 - \frac{9}{2}p_2^2 + \frac{5}{2}p_2 \end{array} \\ D_2(p_2) = \begin{array}{c} (9P_3 - 9p_1 + 9\alpha - \frac{13}{2})p_2 - \frac{9}{2}p_2^2 - \frac{9}{2}\alpha^2 - 9\alpha P_3 + \frac{7}{9} \\ +9\alpha p_1 + \frac{13}{2}\alpha - \frac{9}{2}P_3^2 + 9P_3 p_1 + 2P_3 - \frac{9}{2}p_1^2 + \frac{5}{2}p_1 \end{array} \\ D_3(P_3) = \frac{9}{2}P_3^2 - (9p_1 + 9p_2 - 9\alpha + 2)\, P_3 + \frac{1}{18}\,(9p_1 + 9p_2 - 9\alpha + 2)^2 \end{cases} \tag{24}$$

From D_1, D_2, D_3 in (24) we derive first order conditions for p_1, p_2, P_3 (see (51) in the appendix) which yield a NE of γ_α^3 for $\alpha \le \frac{26}{77}$. For this case, the equilibrium partition of consumers among the available systems is described by Figure 17. In order to reduce the cluttering in Figures 17 and 18, we point out only a small number of vertices in Figure 17 with the purpose to clarify the difference between the two figures;[33] more details are given in the proof of Lemma 7.

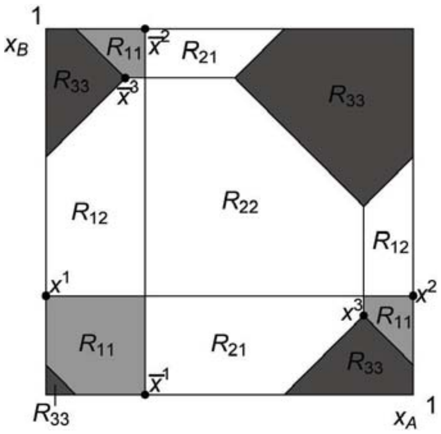


Figure 17

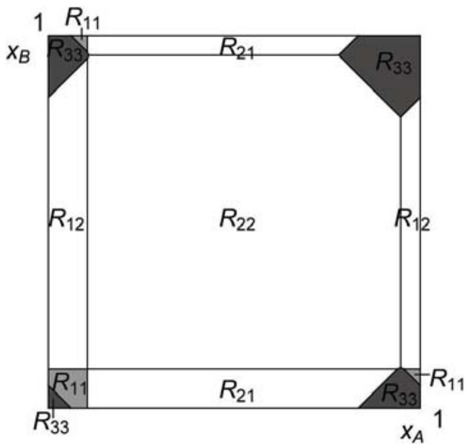


Figure 18

[33] In Figure 17, $\mathbf{x}^1 = (0, \frac{1}{3} - \frac{3}{2}(p_1^* - p_2^* + \alpha))$, $\mathbf{x}^2 = (1, \frac{1}{3} - \frac{3}{2}(p_1^* - p_2^* + \alpha))$, $\mathbf{x}^3 = (\frac{5}{6} + \frac{3}{4}(p_1^* - p_2^* + \alpha), \frac{9}{4}p_1^* - \frac{3}{2}P_3^* + \frac{3}{4}(p_2^* - \alpha) + \frac{1}{6})$.

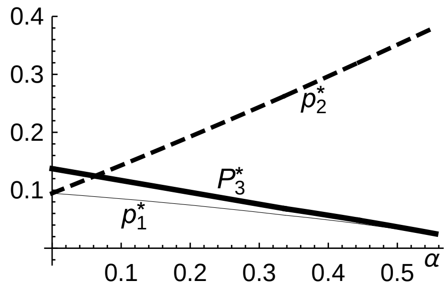
When α is larger than $\frac{26}{77}$, the prices derived from the first order conditions which descend from (24) are not a NE of γ_α^3 because they imply that $\mathbf{x}^3$ in Figure 17 lies above the horizontal segment connecting $\mathbf{x}^1$ to $\mathbf{x}^2$ (likewise, $\bar{\mathbf{x}}^3$ lies to the right of the vertical segment connecting $\bar{\mathbf{x}}^1$ to $\bar{\mathbf{x}}^2$); formally, this means that the equilibrium prices violate the inequality $p_1 \leq \frac{2}{5}P_3 + \frac{1}{5}(p_2 - \alpha) + \frac{2}{45}$. This suggests to derive the expressions of the demand functions when $p_1 > \frac{2}{5}P_3 + \frac{1}{5}(p_2 - \alpha) + \frac{2}{45}$, and then we obtain

$$\begin{cases} D_1(p_1) = & \frac{981}{32}p_1^2 - \left(\frac{153}{8}P_3 + \frac{369}{16}(p_2-\alpha) + \frac{77}{8}\right)p_1 - \frac{99}{32}\alpha^2 - \frac{117}{8}\alpha P_3 + \frac{61}{72} \\ & +\frac{99}{16}\alpha p_2 - \frac{25}{8}\alpha + \frac{9}{8}P_3^2 + \frac{117}{8}P_3 p_2 + \frac{13}{4}P_3 - \frac{99}{32}p_2^2 + \frac{25}{8}p_2 \\ D_2(p_2) = & \left(\frac{153}{16}\alpha + \frac{63}{8}P_3 - \frac{99}{16}p_1 - \frac{53}{8}\right)p_2 - \frac{153}{32}p_2^2 - \frac{153}{32}\alpha^2 - \frac{63}{8}\alpha P_3 + \frac{55}{72} \\ & +\frac{99}{16}\alpha p_1 + \frac{53}{8}\alpha - \frac{45}{8}P_3^2 + \frac{117}{8}P_3 p_1 + \frac{7}{4}P_3 - \frac{369}{32}p_1^2 + \frac{25}{8}p_1 \\ D_3(P_3) = & \frac{9}{4}P_3^2 + \left(\frac{9}{4}p_1 - \frac{45}{4}(p_2-\alpha) - \frac{5}{2}\right)P_3 - \frac{153}{16}p_1^2 + \frac{63}{16}\alpha^2 - \frac{117}{8}\alpha p_1 + \frac{7}{36} \\ & -\frac{63}{8}\alpha p_2 - \frac{7}{4}\alpha - \frac{153}{16}p_1^2 + \frac{117}{8}p_1 p_2 + \frac{13}{4}p_1 + \frac{63}{16}p_2^2 + \frac{7}{4}p_2 \end{cases} \tag{25}$$

From (25) we derive first order conditions for p_1, p_2, P_3 (see (53) in the appendix) which yield a NE of γ_α^3 for $\alpha \in (\frac{26}{77}, \frac{5}{9})$. In this case the consumers partition among the available systems as described in Figure 18.

For subgame γ_α^3 (unlike for γ_α^2) we do not have a closed form expression for the equilibrium prices because the system of the first order conditions is highly non-linear and cannot be solved in closed form. For this reason we resort to a numerical approach, but Lemma 7 verifies that the solution we obtain numerically constitutes a NE of γ_α^3. Figure 19 plots the equilibrium prices

Figure 19
Equilibrium prices in γ_α^3
p_1^* (thin), p_2^* (dashed), P_3^* (thick)



Lemma 7 *In game Γ_α, consider the subgame γ_α^3 which is entered if only firm 3 chooses PB at stage one.*
(i) The demand functions in (24) yield first order conditions that identify a NE of γ_α^3 for $\alpha \in (0, \frac{26}{77}]$.
(ii) The demand functions in (25) yield first order conditions that identify a NE of γ_α^3 for $\alpha \in (\frac{26}{77}, \frac{5}{9})$.

5.4 Bundling by all firms: Subgame γ_α^{123}

In subgame γ_α^{123}, all the three firms bundle. Therefore this game is similar to γ^{123} mentioned in Subsection 3.2, with the only difference that firm 2 offers products with higher quality than the other firms' products. Since firms 1 and 3 are in a symmetric position, we focus on NE of γ_α^{123} in which firms 1 and 3 charge a same price for their bundles. Using P to denote the common value of P_1 and P_3, we define $\delta = \frac{1}{2}P_2 - \frac{1}{2}P - \alpha$.

The demand for firm 2, $D_2(P_2)$, is the area of the set of (x_A, x_B) which satisfy

$$C_{22}(x_A, x_B) < \min\{C_{11}(x_A, x_B), C_{33}(x_A, x_B)\} \tag{26}$$

and it is possible to see that (26) holds for each (x_A, x_B) if $\delta < -\frac{2}{9}$; hence $D_2(P_2) = 1$ in this case. Conversely, if $\delta \geq \frac{1}{9}$ then (26) is violated for each (x_A, x_B) and $D_2(P_2) = 0$. In the intermediate case of $\delta \in [-\frac{2}{9}, -\frac{1}{18})$, the set of (x_A, x_B) such that (26) holds is the convex decagon in Figure 20,[34] with area $1 - 15(\frac{2}{9} + \delta)^2$.

The set of consumers that buy S_{22} in γ_α^{123} given that $\delta = \frac{1}{2}P_2 - \frac{1}{2}P - \alpha$ is between $-\frac{2}{9}$ and $-\frac{1}{18}$ (Figure 20), or δ is between $-\frac{1}{18}$ and $\frac{1}{9}$ (Figure 21) :

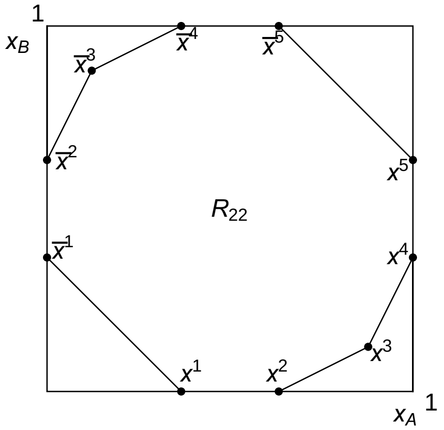


Figure 20

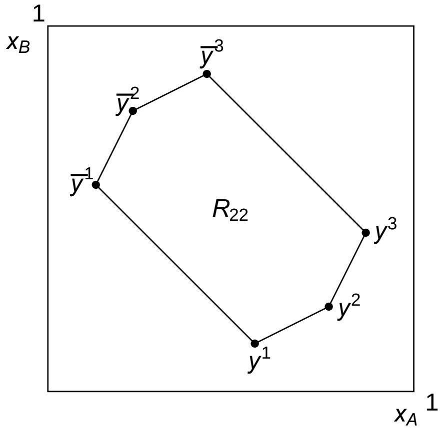


Figure 21

If $\delta \in [-\frac{1}{18}, \frac{1}{9})$, then the set of (x_A, x_B) which satisfy (26) is the hexagon in Figure 21,[35] with area $3(1 - 3\delta)(\frac{1}{9} - \delta)$. Since $\delta = \frac{1}{2}P_2 - \frac{1}{2}P - \alpha$, the demand function for firm 2 can be written as function of P_2:

$$D_2(P_2) = \begin{cases} 1 & \text{if } P_2 < P + 2\alpha - \frac{4}{9} \\ 1 - 15(\frac{2}{9} + \frac{1}{2}P_2 - \frac{1}{2}P - \alpha)^2 & \text{if } P + 2\alpha - \frac{4}{9} \le P_2 < P + 2\alpha - \frac{1}{9} \\ \frac{1}{12}\left(9P + 18\alpha - 9P_2 + 2\right)\left(3P + 6\alpha - 3P_2 + 2\right) & \text{if } P + 2\alpha - \frac{1}{9} \le P_2 \le P + 2\alpha + \frac{2}{9} \\ 0 & \text{if } P + 2\alpha + \frac{2}{9} \le P_2 \end{cases} \tag{27}$$

From (27) we derive P_2 that maximizes firm 2's profit $P_2 D_2(P_2)$:

$$br_2(P, \gamma_\alpha^{123}) = \begin{cases} \frac{2}{3}P + \frac{4}{3}\alpha + \frac{8}{27} - \frac{1}{27}\sqrt{81(P + 2\alpha)^2 + 72(P + 2\alpha) + 28} & \text{if } P + 2\alpha \le \frac{31}{90} \\ \frac{2}{3}P + \frac{4}{3}\alpha - \frac{8}{27} + \frac{1}{27}\sqrt{81(P + 2\alpha)^2 - 72(P + 2\alpha) + \frac{404}{5}} & \text{if } P + 2\alpha > \frac{31}{90} \end{cases} \tag{28}$$

The demand for firm 3 is the area of the set of (x_A, x_B) which satisfy

$$C_{33}(x_A, x_B) < \min\{C_{11}(x_A, x_B), C_{22}(x_A, x_B)\} \tag{29}$$

We define $\mu = P_3 - P_1$, $\theta = P_3 - P_2 + 2\alpha$, and consider P_3 close to P_1, that is μ close to 0. First we examine the case of α close to zero, which implies that P_3 is close to P_2 in equilibrium, therefore also θ is close to zero. Then the set of (x_A, x_B) which satisfy (29) is the union of the three convex sets in Figure 22: the two

[34] In Figure 20, $\mathbf{x}^1 = (\frac{2}{3} + 3\delta, 0)$, $\mathbf{x}^2 = (\frac{1}{3} - 3\delta, 0)$, $\mathbf{x}^3 = (\frac{7}{9} - \delta, \frac{2}{9} + \delta)$, $\mathbf{x}^4 = (1, \frac{2}{3} + 3\delta)$, $\mathbf{x}^5 = (1, \frac{1}{3} - 3\delta)$.

[35] In Figure 21, $\mathbf{y}^1 = (\frac{5}{9} + \delta, \frac{1}{9} + 2\delta)$, $\mathbf{y}^2 = (\frac{7}{9} - \delta, \frac{2}{9} + \delta)$, $\mathbf{y}^3 = (\frac{8}{9} - 2\delta, \frac{4}{9} - \delta)$.

quadrilaterals with vertices $\mathbf{x}^1, ..., \mathbf{x}^4$ and $\bar{\mathbf{x}}^1, ..., \bar{\mathbf{x}}^4$, and the hexagon with vertices $\mathbf{x}^5, \mathbf{x}^6, ..., \bar{\mathbf{x}}^5$:[36]

The set of consumers that buy S_{33} in γ_α^{123} given that $\mu = P_3 - P_1$ is close to 0 and $\theta = P_3 - P_2 + 2\alpha$ close to 0 (Figure 22), or θ is greater than $\frac{1}{9}$ (Figure 23) :

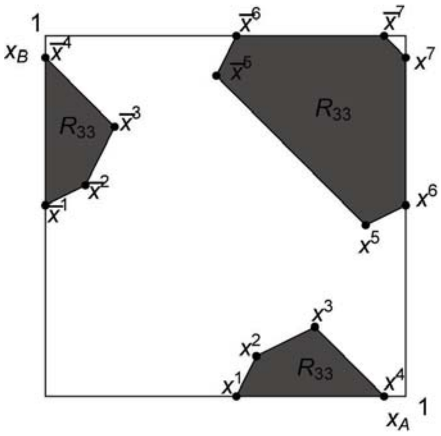


Figure 22

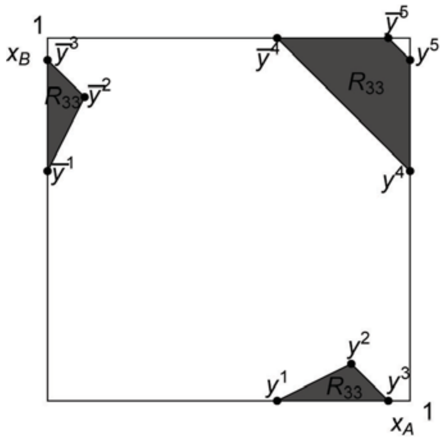


Figure 23

This is a disconnected set with area equal to

$$\begin{aligned} D_3(P_3) &= \frac{9}{4}P_3^2 + \left(\frac{9}{2}\alpha - \frac{9}{4}P_1 - \frac{9}{4}P_2 - 2\right)P_3 - 9P_1\alpha + \frac{9}{2}P_1P_2 \\ &\quad -\frac{9}{8}P_1^2 - \frac{9}{8}P_2^2 + P_1 - \frac{9}{2}\alpha^2 + \frac{9}{2}\alpha P_2 - 2\alpha + P_2 + \frac{1}{3} \end{aligned} \tag{30}$$

Precisely, this expression for $D_3(P_3)$ applies as long as $\frac{1}{2}\mu + \frac{1}{9} \geq \theta$,[37] and from it we obtain a first order condition for P_3 which together with (28) (for the case of $P + 2\alpha \leq \frac{31}{90}$) yields prices P^*, P_2^*. Lemma 8(i) proves that these prices constitute a NE of γ_α^{123} if and only if $\alpha \leq \frac{71}{630}$, which corresponds to the condition $\theta \leq \frac{1}{9}$ (or equivalently $\delta \geq -\frac{1}{18}$).

For the case of $\alpha > \frac{71}{630}$, we still consider μ close to 0 but θ greater than $\frac{1}{9}$. Then the set of (x_A, x_B) which satisfy (29) is the union of the three convex sets in Figure 23: the two triangles with vertices $\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3$ and $\bar{\mathbf{y}}^1, \bar{\mathbf{y}}^2, \bar{\mathbf{y}}^3$, and the quadrilateral with vertices $\mathbf{y}^4, \mathbf{y}^5, \bar{\mathbf{y}}^5, \bar{\mathbf{y}}^4$.[38] The area of this set is

$$\begin{aligned} D_3(P_3) &= 3P_3^2 + \left(\frac{21}{2}\alpha - \frac{3}{4}P_1 - \frac{21}{4}P_2 - \frac{7}{3}\right)P_3 - \frac{3}{8}P_1^2 - 3P_1\alpha \\ &\quad +\frac{3}{2}P_1P_2 + \frac{2}{3}P_1 + \frac{15}{2}\alpha^2 - \frac{15}{2}\alpha P_2 - \frac{10}{3}\alpha + \frac{15}{8}P_2^2 + \frac{5}{3}P_2 + \frac{10}{27} \end{aligned} \tag{31}$$

and it delivers a first order condition for P_3 which together with (28) (for the case of $P + 2\alpha > \frac{31}{90}$) yields the equilibrium prices in Lemma 8(ii).

Lemma 8 *In game Γ_α, consider the subgame γ_α^{123} which is entered if at least two firms choose PB at stage one. In γ_α^{123} there exists a NE with $P_1 = P_3 = P^*$ and $P_2 = P_2^*$ such that*
(i) if $\alpha \leq \frac{71}{630}$, then P^*, P_2^* satisfy the first order condition derived from (30) and $P_2^* = br_2(P^*, \gamma_\alpha^{123})$;
(ii) if $\frac{71}{630} < \alpha < \frac{5}{9}$, then $P^* = \frac{5}{117} - \frac{5}{26}\alpha + \frac{1}{1638}\sqrt{99\,225\alpha^2 - 44\,100\alpha + 29\,470}$ and $P_2^* = \frac{33}{26}\alpha - \frac{11}{39} + \frac{19}{8190}\sqrt{99\,225\alpha^2 - 44\,100\alpha + 29\,470}$.

[36] In Figure 22, $\mathbf{x}^1 = (\frac{1}{2} + \frac{3}{4}\mu, 0)$, $\mathbf{x}^2 = (\frac{5}{9} + \mu - \frac{1}{2}\theta, \frac{1}{9} + \frac{1}{2}\mu - \theta)$, $\mathbf{x}^3 = (\frac{7}{9} + \frac{1}{2}\theta - \mu, \frac{2}{9} - \frac{1}{2}\mu - \frac{1}{2}\theta)$, $\mathbf{x}^4 = (1 - \frac{3}{2}\mu, 0)$ and $\mathbf{x}^5 = (\theta - \frac{1}{2}\mu + \frac{8}{9}, \frac{1}{2}\theta + \frac{1}{2}\mu + \frac{4}{9})$, $\mathbf{x}^6 = (1, \frac{3}{4}\mu + \frac{1}{2})$, $\mathbf{x}^7 = (1, 1 - \frac{3}{2}\mu)$.

[37] Otherwise the vertical coordinate of $\mathbf{x}^2$ (the horizontal coordinate of $\bar{\mathbf{x}}^2$) in Figure 22 is negative, and the horizontal coordinate of $\mathbf{x}^5$ (the vertical coordinate of $\bar{\mathbf{x}}^5$) is greater than 1.

[38] In Figure 23, $\mathbf{y}^1 = (\frac{3}{2}\theta + \frac{1}{3}, 0)$, $\mathbf{y}^2 = (\frac{7}{9} + \frac{1}{2}\theta - \mu, \frac{2}{9} - \frac{1}{2}\mu - \frac{1}{2}\theta)$, $\mathbf{y}^3 = (1 - \frac{3}{2}\mu, 0)$ and $\mathbf{y}^4 = (1, \frac{3}{2}\theta + \frac{1}{3})$, $\mathbf{y}^5 = (1 - \frac{3}{2}\mu, 1)$.

Figures 24 and 25 describe the equilibrium partition of consumers among S_{11}, S_{22}, S_{33} for the two cases of $\alpha \le \frac{71}{630}$ and $\frac{71}{630} < \alpha < \frac{5}{9}$:[39]

Equilibrium partition of consumers in γ_α^{123} among S_{11}, S_{22}, S_{33} when $\alpha \le \frac{71}{630}$ (Figure 24), or when $\alpha \in (\frac{71}{630}, \frac{5}{9})$ (Figure 25) :

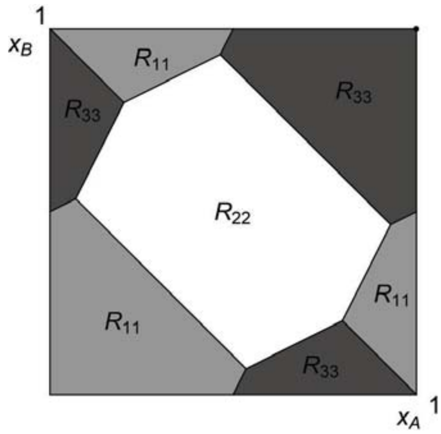


Figure 24

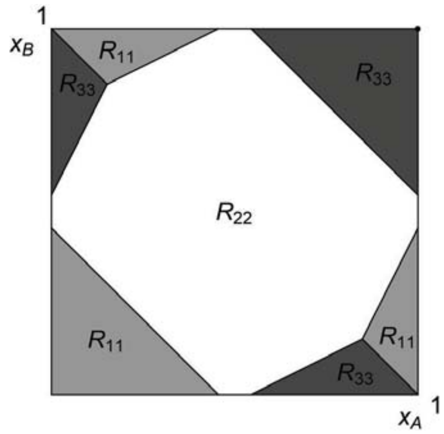


Figure 25

5.5 The marketing stage

Here we study the firms' marketing choices at stage one of Γ_α. Thus, we consider the reduced game with simultaneous moves in which, for each $(a_1, a_2, a_3) \in \{\text{IP,PB}\}^3$, the firms' profits given (a_1, a_2, a_3) are equal to the profits in the stage two subgame which is determined by (a_1, a_2, a_3): see Lemmas 5-8. Precisely, we denote with $\Pi_i^\varnothing$ the profit of firm i in $\gamma_\alpha^\varnothing$ (i.e., $a_1 = a_2 = a_3 =$IP), with Π_i^j the profit of firm i in γ_α^j (i.e., $a_j =$PB, $a_k = a_h =$IP), and with Π_i^{123} the profit of firm i in γ_α^{123} (i.e., at least two firms have chosen PB). In the normal form below, firm 1 chooses a row, firm 2 chooses a column, firm 3 chooses a matrix.

$a_3 =$ IP

	$a_2 =$ IP	$a_2 =$ PB
$a_1 =$ IP	$\Pi_1^\varnothing, \Pi_2^\varnothing, \Pi_3^\varnothing$	$\Pi_1^2, \Pi_2^2, \Pi_3^2$
$a_1 =$ PB	$\Pi_1^1, \Pi_2^1, \Pi_3^1$	$\Pi_1^{123}, \Pi_2^{123}, \Pi_3^{123}$

$a_3 =$ PB

	$a_2 =$ IP	$a_2 =$ PB
$a_1 =$ IP	$\Pi_1^3, \Pi_2^3, \Pi_3^3$	$\Pi_1^{123}, \Pi_2^{123}, \Pi_3^{123}$
$a_1 =$ PB	$\Pi_1^{123}, \Pi_2^{123}, \Pi_3^{123}$	$\Pi_1^{123}, \Pi_2^{123}, \Pi_3^{123}$

We have mentioned in Section 4 that (PB,PB,PB) is a NE for each α, hence in the following we examine the existence of other NE. To this purpose, it is useful to compare:

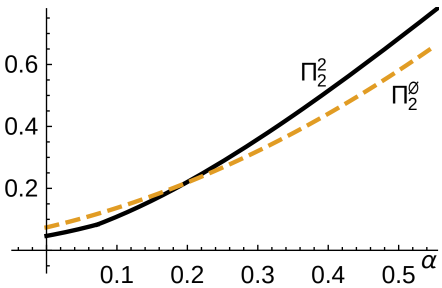
- Π_2^2 with $\Pi_2^\varnothing$ and Π_2^{123} with Π_2^3, in order to inquire the dominant firm's incentives to bundle when no other firm bundles, or when only one other firm bundles.
- Π_3^3 with $\Pi_3^\varnothing$, Π_3^{123} with Π_3^1, and Π_3^{123} with Π_3^2 in order to learn the incentive to bundle of a dominated firm when no other firm bundles, or only the other dominated firm bundles, or only the dominant firm bundles.

We use Lemmas 5-8 and numeric analysis to perform the above comparisons. For instance, Figure 26 below plots Π_2^2 and $\Pi_2^\varnothing$ as a function of α (the appendix includes the plots of the profit functions involved

[39] The vertices in Figure 24 (Figure 25) are obtained from the vertices in Figures 21 and 22 (20 and 23), given the equilibrium prices; notice that $\theta = -2\delta$ at the equilibrium prices.

in the other comparisons)

Figure 26
Π_2^2 (solid) vs $\Pi_2^{\varnothing}$ (dashed)



As a result we conclude that

$$\Pi_2^2 < \Pi_2^{\varnothing} \quad \text{if} \quad \alpha \in (0, \alpha'), \qquad \Pi_2^2 > \Pi_2^{\varnothing} \quad \text{if} \quad \alpha \in (\alpha', \frac{5}{9}), \qquad \text{with } \alpha' = 0.1953 \tag{32}$$

$$\Pi_2^{123} < \Pi_2^3 \quad \text{if} \quad \alpha \in (0, 0.1709), \qquad \Pi_2^{123} > \Pi_2^3 \quad \text{if} \quad \alpha \in (0.1709, \frac{5}{9}) \tag{33}$$

From (32)-(33) we deduce that (i) if $\alpha < 0.1709$, then firm 2 plays IP in any NE (neglecting the NE (PB,PB,PB)); (ii) if $\alpha > \alpha'$, then PB is weakly dominant for firm 2; (iii) if α is between 0.1709 and α', then firm 2's best reply is IP if $a_1 = a_3 =$IP, is PB if $a_1 =$PB or $a_3 =$PB.

The profit comparisons for firm 3 reveal that

$$\Pi_3^3 < \Pi_3^{\varnothing} \quad \text{if} \quad \alpha \in (0, 0.496), \qquad \Pi_3^3 > \Pi_3^{\varnothing} \quad \text{if} \quad \alpha \in (0.496, \frac{5}{9}) \tag{34}$$

$$\Pi_3^1 > \Pi_3^{123} \quad \text{for each} \quad \alpha \in (0, \frac{5}{9}) \tag{35}$$

$$\Pi_3^{123} < \Pi_3^2 \text{ if } \alpha \in (0, 0.051) \cup (0.1, \alpha''), \;\; \Pi_3^{123} > \Pi_3^2 \text{ if } \alpha \in (0.051, 0.1) \cup (\alpha'', \frac{5}{9}), \;\; \text{with } \alpha'' = 0.1981 \tag{36}$$

From (34)-(36) we see that firm 3 never wants to bundle if $a_1 =$PB, $a_2 =$IP, but wants to bundle if $a_1 = a_2 =$IP and $\alpha > 0.496$, or if $a_1 =$IP, $a_2 =$PB and α is between 0.051 and 0.1, or is greater than α''.

With these informations we can identify the NE of the reduced game, distinguishing three intervals for α: $(0, \alpha')$, (α', α''), $(\alpha'', \frac{5}{9})$.

In the interval $(0, \alpha')$, firm 2 may want to play PB only if at least one of the other firms plays PB and $\alpha > 0.1709$. However, for $\alpha < \alpha'$ firm 3 (firm 1) has an incentive to play PB only if $a_1 =$IP, $a_2 =$PB and $\alpha \in (0.051, 0.1)$. Hence (IP,IP,IP) is the unique NE different from (PB,PB,PB). This extends Proposition 1, which covers the case of $\alpha = 0$.

We obtain different results when $\alpha \in (\alpha', \alpha'')$, because then firm 2 wants to bundle even if no other firm does so. However, (36) reveals that firm 3 (firm 1) does not want to bundle if only firm 2 bundles. Hence (IP,IP,IP) is not a NE, but (IP,PB,IP) is.

Finally, for $\alpha \in (\alpha'', \frac{5}{9})$ there is a change in the preference of firm 3 (firm 1): now firm 3 wants to bundle given that only firm 2 bundles. Hence, in each NE at least two firms bundle and each NE is equivalent to (PB,PB,PB). Next proposition summarizes these results.

Proposition 2 *In the reduced game for Γ_α, (PB, PB, PB) is a NE for each α and*
(i) when $\alpha \in (0, \alpha')$, there exists a unique other NE, (IP, IP, IP);
(ii) when $\alpha \in (\alpha', \alpha'')$, there exists a unique other NE, (IP, PB, IP);
(iii) when $\alpha \in (\alpha'', \frac{5}{9})$, each other NE is equivalent to (PB, PB, PB).

Unlike Proposition 1, Proposition 2 establishes that in the asymmetric setting we are considering, firms sometimes have individual incentives to bundle. These incentives depend on the magnitude of the asymmetry and on the marketing choices of the other firms. Precisely, the difference between Proposition 1 and Proposition 2 lies in Proposition 2(ii–iii), which is determined by the inequalities $\Pi_2^2 > \Pi_2^{\varnothing}$ and $\Pi_3^{123} > \Pi_3^2$ in (32) and (36). In the rest of this section we explore why these inequalities are satisfied for $\alpha > \alpha'$ and for $\alpha > \alpha''$, respectively, even though they fail to hold when $\alpha = 0$.

Bundling is profitable for firm 2 when no other firm bundles and $\alpha > \alpha'$ When α is equal to 0, we have explained in Section 4 that PB reduces the profit of firm 2 (given $a_1 = a_3 =$IP) because of a negative demand size effect and because the demand elasticity effect makes price competition fiercer. However, a different result emerges if $\alpha > 0$ is not small. Starting with the demand size effect, we consider γ_α^2 with $p_1 = p_3 = p^*$, $P_2 = 2p_2^*$ (p^*, p_2^* are the NE prices in $\gamma_\alpha^{\varnothing}$: see (16)). As in Subsection 4, we focus on the set $R_{32}^{\varnothing}$ of consumers that buy S_{32} in $\gamma_\alpha^{\varnothing}$,[40] which is the rectangle $[\frac{2}{3} + \frac{3}{5}\alpha, 1) \times [\frac{1}{3} - \frac{3}{5}\alpha, \frac{2}{3} + \frac{3}{5}\alpha)$ shown in Figure 27 with three dashed edges. Since S_{32} is unavailable in γ_α^2, each consumer in $R_{32}^{\varnothing}$ will buy one system among S_{22}, S_{33}, S_{31}. With respect to S_{32}, all these systems reduce a consumer's utility, but simple algebra shows that the utility decrease with the choice of S_{22} is decreasing in α, whereas the utility decrease with S_{33} or S_{31} is increasing in α, in such a way that more than half of the consumers in $R_{32}^{\varnothing}$ buy S_{22} if $\alpha > \frac{5}{36}$. Essentially, when $\alpha > \frac{5}{36}$ for a majority of consumers in $R_{32}^{\varnothing}$ it is not convenient to give up product B_2, even though that requires to buy A_2 which they like less than A_3. Figure 27 also represents the set R_{22} of consumers that buy S_{22} in γ_α^2 when $\alpha > \frac{5}{36}$ and $p_1 = p_3 = p^*$, $P_2 = 2p_2^*$.

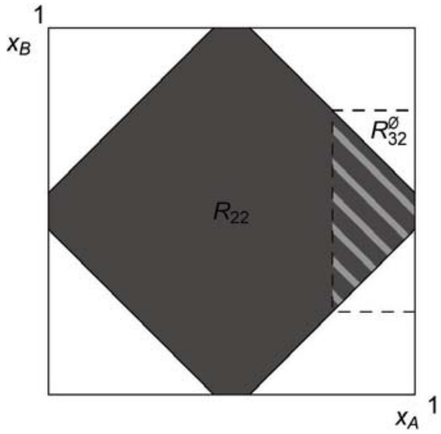


Figure 27:
The set $R_{32}^{\varnothing}$ (with dashed edges) of the consumers that in $\gamma_\alpha^{\varnothing}$ buy S_{32}, and the set R_{22} (with solid edges) of consumers that buy S_{22} in γ_α^2 given $p_1 = p_3 = p^*$, $P_2 = 2p_2^*$ (p^*, p_2^* are NE prices in $\gamma_\alpha^{\varnothing}$: see (16))

Therefore the demand size effect for firm 2 is negative if $\alpha < \frac{5}{36}$ but is positive if $\alpha > \frac{5}{36}$ (compare Figure 12 with Figure 27). We also remark that this effect is weak if α is close to $\frac{5}{9}$, as then the market share of firm 2 is already very large in $\gamma_\alpha^{\varnothing}$, hence the set $R_{32}^{\varnothing}$ is small.

Also the demand elasticity effect depends on α: given $p_1 = p_3 = p^*$, we find that at $P_2 = 2p_2^*$, D_2 is elastic if $\alpha < 0.3115$, is inelastic if $\alpha > 0.3115$. In Section 4 we have mentioned the first case when $\alpha = 0$; the second case is simple to see when $\alpha = \frac{5}{9}$, as then the equilibrium prices for $\gamma_\alpha^{\varnothing}$ are $p_1^* = p_3^* = 0$, $p_2^* = \frac{1}{3}$ and all consumers buy both products from firm 2. Then, from (18), $D_2(P_2) = 1 - \frac{1}{2}\left(3P_2 - 2\right)^2$ if P_2 is equal to or slightly larger than $2p_2^* = \frac{2}{3}$ and $D_2'(\frac{2}{3}) = 0$, which means that D_2 is very inelastic at $P_2 = \frac{2}{3}$: if P_2 increases by a small $\Delta P_2 > 0$ above $\frac{2}{3}$, then the reduction in demand for S_{22} is of order $(\Delta P_2)^2$.[41]

[40] Similar arguments apply to the sets $R_{12}^{\varnothing}, R_{21}^{\varnothing}, R_{23}^{\varnothing}$, and there is no change in the purchases of consumers that in $\gamma_\alpha^{\varnothing}$ buy no product of firm 2, or both products of firm 2.

[41] This is because the sets of consumers in $[0,1) \times [0,1)$ firm 2 loses because of the increase in P_2 are four right triangles at the corners of $[0,1) \times [0,1)$, each with edges proportional to ΔP_2. Hence their combined area is proportional to $(\Delta P_2)^2$.

The incentive of firm 2 to reduce P_2 for $\alpha < 0.3115$ has the effect of inducing firms 1, 3 to reduce their prices, as we remarked for the case of $\alpha = 0$. Moreover, except for values of α close to zero, firms 1, 3 have incentive to reduce p_1, p_3 below p^* even if $P_2 = 2p_2^*$. This harms firm 2 and has the effect that $\alpha > \frac{5}{36}$ does not suffice to make firm 2 prefer γ_α^2 to $\gamma_\alpha^\varnothing$; indeed (32) reveals that $\Pi_2^2 > \Pi_2^\varnothing$ holds if and only if $\alpha > \alpha'$.

As a final remark, notice that for α larger than 0.4395, firms 1, 3 prefer γ_α^2 to $\gamma_\alpha^\varnothing$, that is $\Pi_3^2 > \Pi_3^\varnothing$. This is immediate when $\alpha = \frac{5}{9}$ since then $\Pi_3^\varnothing = 0$, and in γ_α^2 firm 2 wants to increase P_2 above $2p_2^*$ because D_2 is very inelastic. The increase in P_2 implies that firm 3 (firm 1) earns a positive market share and profit if $p_3 > 0$ ($p_1 > 0$) is close to 0; hence $\Pi_3^2 > \Pi_3^\varnothing = 0$.[42] The inequality $\Pi_3^2 > \Pi_3^\varnothing$ is satisfied also for α smaller than (but close to) $\frac{5}{9}$, as in this case there is a demand size effect which is negative for firms 1, 3, but is weak and is dominated by the demand elasticity effect that induces firm 2 to increase P_2 above $2p_2^*$.

Bundling is profitable for firm 3 if firm 2 bundles and $\alpha > \alpha''$ In Section 4 we have explained why firm 3 prefers IP to PB given a_1 =IP, a_2 =PB when $\alpha = 0$. Now we illustrate why the opposite holds, that is $\Pi_3^{123} > \Pi_3^2$, when $\alpha > \alpha''$.

We first notice that the demand size effect is negative for firm 3. Comparing the NE of γ_α^2 with the outcome in γ_α^{123} given $P_1 = P_3 = 2p^*$, $P_2 = P_2^*$ (p^*, P_2^* are the NE prices of γ_α^2: see Lemma 6 and (23)) reveals that in the latter case the market share and profit of firm 3 is reduced. In order to see why, notice that $\alpha'' > \frac{13}{180}$, therefore in γ_α^2 the equilibrium partition of consumers is described by Figure 15. Moving to γ_α^{123} with unchanged prices makes S_{13}, S_{31} unavailable, and firm 3's profit derives only from the sale of S_{33}. The consumers that buy S_{33} in the NE of γ_α^2 continue to buy S_{33} in γ_α^{123}, hence the demand size effect is determined by the purchases of the consumers that buy S_{31} in γ_α^2 (similar arguments apply to S_{13}); let R_{31}^2 denote this set. In γ_α^{123}, suppose for one moment that S_{22} is not available. Then the consumers in R_{31}^2 split equally between S_{11} and S_{33} since $P_1 = P_3$ and one half of them is closer to S_{11}, the other half is closer to S_{33}. However, the presence of S_{22} has a relevant effect since for the consumers in R_{31}^2, S_{11} and S_{33} are inferior to S_{31} but the consumers at the border between R_{31}^2 and R_{22}^2 (see Figure 15) are actually indifferent between S_{22} and S_{31}. Hence, the consumers in R_{31}^2 close to this border prefer S_{22} to both S_{11} and S_{33} and less than one half of the consumers in R_{31}^2 buy S_{33}; this makes the demand size effect negative for firm 3. Therefore, moving from γ_α^2 to γ_α^{123} with unchanged prices improves the situation of firm 2 by reducing the competition it faces as S_{13}, S_{31} are not available anymore. This can be seen in Figure 20, in which the union between the two triangles $\mathbf{x}^2\mathbf{x}^3\mathbf{x}^4$ and $\bar{\mathbf{x}}^2\bar{\mathbf{x}}^3\bar{\mathbf{x}}^4$ is the set of consumers firm 2 wins over in γ_α^{123} with respect to γ_α^2.

Nevertheless, also firm 3 (firm 1) prefers γ_α^{123} to γ_α^2 for large α because firm 2 is less aggressive in γ_α^{123} than in γ_α^2. Precisely, suppose that $p_1 = p_3 = p$ in γ_α^2 and $P_1 = P_3 = 2p$ in γ_α^{123}; therefore system S_{11} (S_{33}) has the same price in γ_α^2 as in γ_α^{123}, but in γ_α^2 also the systems S_{13}, S_{31} are available. Comparing (19) with (28) shows that $br_2(2p, \gamma_\alpha^{123}) > br_2(p, \gamma_\alpha^2)$, that is the best reply for firm 2 is greater in γ_α^{123} than in γ_α^2. This occurs because a higher number of inframarginal consumers for firm 2 in γ_α^{123} than in γ_α^2 (due to the positive demand size effect for firm 2) makes it more profitable for firm 2 to increase P_2, and also because a same increase in P_2 in γ_α^2 and in γ_α^{123} leads to a smaller loss of consumers in γ_α^{123} than in γ_α^2 (this is a consequence of how the triangles $\mathbf{x}^2\mathbf{x}^3\mathbf{x}^4$ and $\bar{\mathbf{x}}^2\bar{\mathbf{x}}^3\bar{\mathbf{x}}^4$ in Figure 20 depend on P_2). Although the demand size effect is negative for firm 3, when α is large that effect is weak as the market share of firm 2 is already very large in γ_α^2. From the point of view of firm 3, this effect is dominated by the demand elasticity effect which induces less aggressive pricing by firm 2, and allows firm 3 (firm 1) to increase p_3 (p_1) and earn a higher profit than in γ_α^2. Thus, by reducing the own competitiveness through a reduction of the number of systems, firm 3 (firm 1) increases the own profit because in a less competitive environment firm 2 charges a

[42] This is similar to what happens in Hurkens, Jeon, Menicucci (2018), in a case with two firms for which footnote 7 applies.

higher price.

6 Conclusions

In this paper we have examined an oligopoly model in which some firms, but not all, bundle. Our results provide indications about the firms' individual incentives to bundle, and suggest that such incentives may exist only among asymmetric firms.

Our analysis is carried out under the assumption that firms offer spatially differentiated products according to Salop's circular city model. This implies that the outcome of competition depends both on the number and on the locations of the firms that bundle. An alternative approach is to rely on the random utility model of Perloff and Salop (1985), as in Zhou (2017). In this model any two firms that bundle (any two firms that do not bundle) are in the same situation. Hence, the analysis of each stage two subgame only depends on the number of firms that bundle, and not on their identities.

Another possible line of research consists in analyzing the game with a single stage in which simultaneously each firm chooses bundling or independent sales, and the price for its bundle (in the first case) or the prices for its individual products (in the second case). As we mentioned in the introduction, our model seems appropriate to represent a situation in which the choice between bundling and independent pricing cannot be undone, whereas a single stage game is adequate if a firm can easily switch from one price regime to the other. In such a game there are no subgames, but a firm may alter its pricing regime in response to the strategies of the other firms; thus more powerful deviations may exist than in the game we have analyzed.

7 Appendix

7.1 Proof of Lemma 3

From (11) we derive the following first order condition for p_3, at $p_3 = p_1 = p$:

$$-\frac{9}{2}P_2^2 + 27P_2p + 2P_2 - 36p^2 - \frac{21}{2}p + \frac{7}{9} = 0 \tag{37}$$

Together with $P_2 = \frac{2}{3}p + \frac{2}{27}$ from (9) (given $p < \frac{5}{36}$), (37) identifies the prices in Lemma 3. In this proof we show that such prices constitute a NE for γ^2. We use p^* to denote the common value of p_1^* and p_3^*.

For firm 2, from (9) we know that $P_2^* = br_2(p^*, \gamma^2)$ is a best reply given that $p_1 = p_3 = p^*$.

In the rest of this proof we suppose that firm 1, firm 2 play $p_1 = p^*$, $P_2 = P_2^*$. We derive the complete demand function for firm 3 and prove that playing $p_3 = p^*$ is a best reply for firm 3.

The demand function of firm 3 First we consider $p_3 < p^*$ and argue as in Steps 1 and 2 of Subsection 3.3, assuming initially that S_{22} is not available. The inequality $C_3(x) < C_1(x)$ holds for $x \in [0, y) \cup [z, 1)$, with $y = \frac{3}{2}(p^* - p_3)$, $z = \frac{1}{2} - \frac{1}{2}y$. Therefore the consumers partition among $S_{11}, S_{13}, S_{31}, S_{33}$ as follows:

$$\begin{cases} Q_{11} = [y, z) \times [y, z), & Q_{13}^S = [y, z) \times [0, y), & Q_{13}^N = [y, z) \times [z, 1), \\ Q_{31}^W = [0, y) \times [y, z), & Q_{31}^E = [z, 1) \times [y, z), & Q_{33}^{SW} = [0, y) \times [0, y), \\ Q_{33}^{SE} = [z, 1) \times [0, y), & Q_{33}^{NW} = [0, y) \times [z, 1), & Q_{33}^{NE} = [z, 1) \times [z, 1) \end{cases} \tag{38}$$

see Figure 28:

The sets in (38) (Figure 28) and the sets $R_{33}^{SW}, R_{33}^{SE}, R_{33}^{NW}, R_{33}^{NE}, R_{31}^{W}, R_{31}^{E}, R_{13}^{S}, R_{13}^{N}$ when $p_3 < p^*$ (Figure 29)

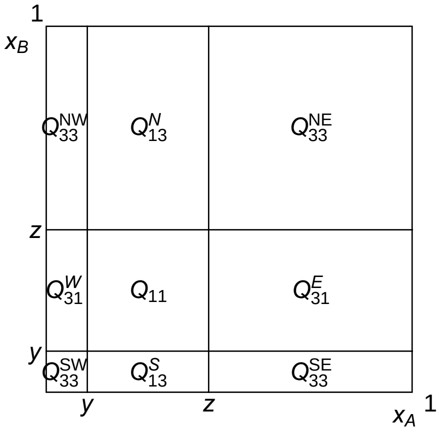


Figure 28

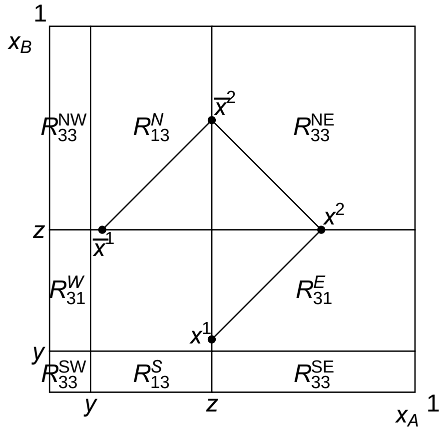


Figure 29

We neglect Q_{11} and solve $C_{22}(x_A, x_B) > C_{ij}(x_A, x_B)$ in Q_{ij} for $ij \neq 11$ to determine the set R_{ij} of consumers that prefers S_{ij} to S_{22}: see Figure 29. We use $\mathcal{A}(R_{ij})$ to denote the area of the set R_{ij}.

In Q_{33}^{SW}, $C_{22}(x_A, x_B) > C_{33}(x_A, x_B)$ reduces to $x_B < \frac{3}{4}P_2^* - \frac{3}{2}p_3 - x_A + \frac{1}{3}$, which is satisfied for each $(x_A, x_B) \in Q_{33}^{SW}$, for each $p_3 < p^*$. Hence $R_{33}^{SW} = Q_{33}^{SW}$ and $\mathcal{A}(R_{33}^{SW}) = y^2$.

In Q_{33}^{SE}, $C_{22}(x_A, x_B) > C_{33}(x_A, x_B)$ reduces to $x_B < \frac{3}{4}P_2^* - \frac{3}{2}p_3 + \frac{1}{2}x_A - \frac{1}{6}$, which holds for each $(x_A, x_B) \in Q_{33}^{SE}$, for each $p_3 < p^*$. Hence $R_{33}^{SE} = Q_{33}^{SE}$ and $\mathcal{A}(R_{31}^{SE}) = y(1-z)$. Likewise, $R_{33}^{NW} = Q_{33}^{NW}$ and $\mathcal{A}(R_{33}^{NW}) = y(1-z)$.

In Q_{33}^{NE}, $C_{22}(x_A, x_B) > C_{33}(x_A, x_B)$ is equivalent to $x_B > \frac{4}{3} + 3p_3 - \frac{3}{2}P_2^* - x_A$. Hence R_{33}^{NE} coincides with Q_{33}^{NE} except for a triangle in the left bottom of Q_{33}^{NE} with vertices (z,z), $\mathbf{x}^2 = (\frac{4}{3} + 3p_3 - \frac{3}{2}P_2^* - z, z)$, $\bar{\mathbf{x}}^2 = (z, \frac{4}{3} + 3p_3 - \frac{3}{2}P_2^* - z)$ (see Figure 29) and $\mathcal{A}(R_{33}) = (1-z)^2 - \frac{1}{2}(\frac{4}{3} + 3p_3 - \frac{3}{2}P_2^* - 2z)^2$.

In Q_{31}^{W}, $C_{22}(x_A, x_B) > C_{31}(x_A, x_B)$ is equivalent to $x_B < \frac{3}{2}P_2^* - \frac{3}{2}p^* - \frac{3}{2}p_3 - 2x_A + \frac{2}{3}$, which holds for each $(x_A, x_B) \in Q_{31}^{W}$, for each $p_3 < p^*$. Hence $R_{31}^{W} = Q_{31}^{W}$ and $\mathcal{A}(R_{31}^{W}) = y(z-y)$. Likewise, $R_{13}^{S} = Q_{13}^{S}$ and $\mathcal{A}(R_{13}^{S}) = y(z-y)$.

In Q_{31}^{E}, $C_{22}(x_A, x_B) > C_{31}(x_A, x_B)$ reduces to $x_B < \frac{3}{2}P_2^* - \frac{1}{3} - \frac{3}{2}p^* - \frac{3}{2}p_3 + x_A$, which makes R_{31}^{E} equal to Q_{31}^{E} minus a triangle in the top left of Q_{31}^{E} with vertices (z,z), $\mathbf{x}^1 = (z, \frac{3}{2}P_2^* - \frac{1}{3} - \frac{3}{2}p^* - \frac{3}{2}p_3 + z)$, $\mathbf{x}^2$ (see Figure 29). Hence $\mathcal{A}(R_{31}^{E}) = (z-y)(1-z) - \frac{1}{2}(\frac{1}{3} + \frac{3}{2}p^* + \frac{3}{2}p_3 - \frac{3}{2}P_2^*)^2$. Likewise, $\mathcal{A}(R_{13}^{N}) = (z-y)(1-z) - \frac{1}{2}(\frac{1}{3} + \frac{3}{2}p^* + \frac{3}{2}p_3 - \frac{3}{2}P_2^*)^2$.

Hence the total demand of firm 3 when $p_3 < p^*$ is

$$\begin{aligned} & 2\mathcal{A}(R_{33}^{SW}) + 2\mathcal{A}(R_{33}^{SE}) + 2\mathcal{A}(R_{33}^{NW}) + 2\mathcal{A}(R_{33}^{NE}) + \mathcal{A}(R_{31}^{W}) + \mathcal{A}(R_{31}^{E}) + \mathcal{A}(R_{13}^{S}) + \mathcal{A}(R_{13}^{N}) \\ = \; & 1 + \frac{9}{2}p^* - \frac{9}{2}p_3 - \frac{9}{2}(\frac{2}{9} + p^* + p_3 - P_2^*)^2 \end{aligned}$$

Now we consider $p_3 > p^*$. Then $C_3(x) < C_1(x)$ holds for $x \in [y, z)$ with $y = \frac{1}{2} + \frac{3}{4}(p_3 - p^*)$, $z = 2 - 2y$, and the consumers partition among $S_{11}, S_{13}, S_{31}, S_{33}$ as follows:

$$\begin{aligned} Q_{11} &= ([0,y) \cup [z,1)) \times ([0,y) \cup [z,1)), \qquad Q_{13}^{W} = [0,y) \times [y,z), \qquad Q_{13}^{E} = [z,1) \times [y,z), \\ Q_{31}^{S} &= [y,z) \times [0,y), \qquad Q_{31}^{N} = [y,z) \times [z,1), \qquad Q_{33} = [y,z) \times [y,z) \end{aligned}$$

Solving $C_{22}(x_A, x_B) > C_{ij}(x_A, x_B)$ in $Q_{13}^{W}, Q_{13}^{E}, Q_{31}^{S}, Q_{31}^{N}, Q_{33}$ yields the sets $R_{13}^{W}, R_{13}^{E}, R_{31}^{S}, R_{31}^{N}, R_{33}$ such that $D_3(p_3) = 2\mathcal{A}(R_{33}) + \mathcal{A}(R_{13}^{W}) + \mathcal{A}(R_{31}^{S}) + \mathcal{A}(R_{13}^{E}) + \mathcal{A}(R_{31}^{N})$. In fact, since $\mathcal{A}(R_{31}^{S}) = \mathcal{A}(R_{13}^{W})$ and $\mathcal{A}(R_{31}^{N}) = \mathcal{A}(R_{13}^{E})$, we can evaluate $D_3(p_3)$ as $2\mathcal{A}(R_{33}) + 2\mathcal{A}(R_{13}^{W}) + 2\mathcal{A}(R_{13}^{E})$. In Q_{33}, $C_{22}(x_A, x_B) > C_{33}(x_A, x_B)$ is equivalent to $x_B > -\frac{3}{2}P_2^* + 3p_3 - x_A + \frac{4}{3}$. In Q_{13}^{W}, $C_{22}(x_A, x_B) > C_{13}(x_A, x_B)$ is equivalent to $x_B > \frac{3}{2}p^* - \frac{3}{2}P_2^* + \frac{3}{2}p_3 + x_A + \frac{1}{3}$. In Q_{13}^{E}, $C_{22}(x_A, x_B) > C_{13}(x_A, x_B)$ is equivalent to $x_B > \frac{3}{2}p^* - \frac{3}{2}P_2^* + \frac{3}{2}p_3 - 2x_A + \frac{7}{3}$.

- If $p^* < p_3 < \frac{2}{45} + \frac{1}{5}p^* + \frac{2}{5}P_2^*$, then the sets R_{33}, R_{13}^W, R_{13}^E, R_{31}^S, R_{31}^N are as in Figure 8 and $D_3(p_3) = 1 + \frac{9}{2}p^* - \frac{9}{2}p_3 - \frac{9}{2}(\frac{2}{9} + p^* + p_3 - P_2^*)^2$ as in (11).
- If $\frac{2}{5}P_2^* + \frac{1}{5}p^* + \frac{2}{45} < p_3 < \frac{1}{4}P_2^* + \frac{1}{2}p^* + \frac{1}{9}$ (that is, if $0.1183 < p_3 < 0.1928$) then the sets R_{33}, R_{13}^W, R_{13}^E, R_{31}^S, R_{31}^N are as in Figure 30,[43] and $D_3(p_3) = \frac{981}{32}p_3^2 - (\frac{153}{8}P_2^* + \frac{369}{16}p^* + \frac{77}{8})p_3 + \frac{9}{8}(P_2^*)^2 + \frac{117}{8}P_2^*p^* + \frac{13}{4}P_2^* - \frac{99}{32}(p^*)^2 + \frac{25}{8}p^* + \frac{61}{72}$.
- If $\frac{1}{4}P_2^* + \frac{1}{2}p^* + \frac{1}{9} < p_3 < 2P_2^* - 3p^* + \frac{2}{9}$ (that is, if $0.1928 < p_3 < 0.2124$) then the sets R_{13}^W, R_{13}^E, R_{31}^S, R_{31}^N are as in Figure 31,[44] and $D_3(p_3) = \frac{405}{32}p_3^2 - (\frac{81}{8}P_2^* + \frac{81}{16}p^* + \frac{45}{8})p_3 + \frac{9}{4}P_2^* + \frac{9}{8}p^* + \frac{81}{8}P_2^*p^* - \frac{243}{32}(p^*)^2 + \frac{5}{8}$.
- If $2P_2^* - 3p^* + \frac{2}{9} < p_3 < \frac{1}{2}P_2^* + \frac{2}{9}$ (that is, if $0.2124 < p_3 < 0.2909$), then the sets R_{13}^W, R_{13}^E, R_{31}^S, R_{31}^N are as in Figure 32,[45] and $D_3(p_3) = \frac{27}{2}(\frac{2}{9} + \frac{1}{2}P_2^* - p_3)^2$.

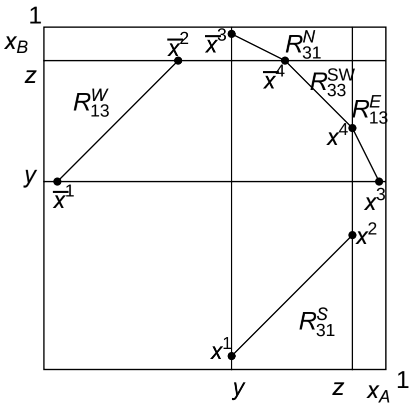


Figure 30

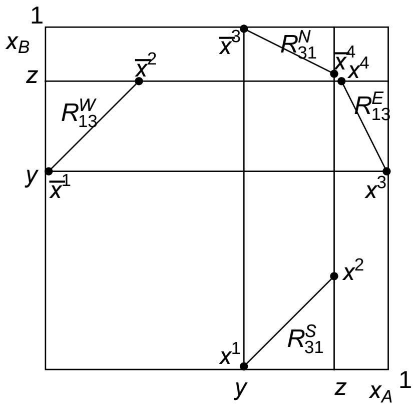


Figure 31

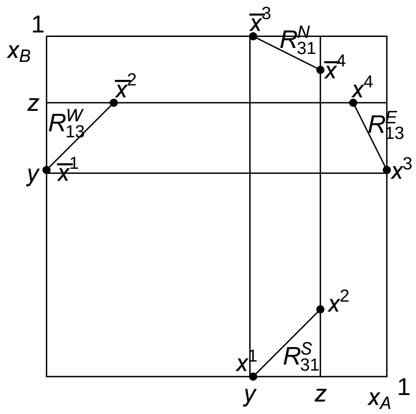


Figure 32

Summarizing, the complete demand function for firm 3 is

$$D_3(p_3) = \begin{cases} 1 + \frac{9}{2}p^* - \frac{9}{2}p_3 - \frac{9}{2}(\frac{2}{9} + p_3 + p^* - P_2^*)^2 & \text{if } 0 \le p_1 \le \frac{1}{5}p^* + \frac{2}{5}P_2^* + \frac{2}{45} \\ \frac{981}{32}p_3^2 - (\frac{153}{8}P_2^* + \frac{369}{16}p^* + \frac{77}{8})p_3 + \frac{9}{8}(P_2^*)^2 & \\ +\frac{117}{8}P_2^*p^* + \frac{13}{4}P_2^* - \frac{99}{32}(p^*)^2 + \frac{25}{8}p^* + \frac{61}{72} & \text{if } \frac{1}{5}p^* + \frac{2}{5}P_2^* + \frac{2}{45} < p_1 \le \frac{1}{2}p^* + \frac{1}{4}P_2^* + \frac{1}{9} \\ (\frac{1}{2} + \frac{9}{4}p^* - \frac{9}{4}p_3)(\frac{5}{4} - \frac{27}{8}p^* + \frac{9}{2}P_2^* - \frac{45}{8}p_3) & \text{if } \frac{1}{2}p^* + \frac{1}{4}P_2^* + \frac{1}{9} < p_1 \le 2P_2^* - 3p^* + \frac{2}{9} \\ \frac{27}{2}(\frac{2}{9} + \frac{1}{2}P_2^* - p_3)^2 & \text{if } 2P_2^* - 3p^* + \frac{2}{9} < p_3 \le \frac{1}{2}P_2^* + \frac{2}{9} \\ 0 & \text{if } p_1 > \frac{1}{2}P_2^* + \frac{2}{9} \end{cases} \tag{39}$$

[43] In Figure 30, $\mathbf{x}^1 = (y, \frac{3}{2}P_2^* - \frac{3}{2}p^* + y - \frac{3}{2}p_3 - \frac{1}{3})$, $\mathbf{x}^2 = (z, \frac{3}{2}P_2^* - \frac{3}{2}p^* + z - \frac{3}{2}p_3 - \frac{1}{3})$, $\mathbf{x}^3 = (\frac{3}{4}p^* - \frac{3}{4}P_2^* - \frac{1}{2}y + \frac{3}{4}p_3 + \frac{7}{6}, y)$, $\mathbf{x}^4 = (z, -\frac{3}{2}P_2^* + 3p_3 - z + \frac{4}{3})$.

[44] In Figure 31, $\mathbf{x}^1 = (y, \frac{3}{2}P_2^* - \frac{3}{2}p^* + y - \frac{3}{2}p_3 - \frac{1}{3})$, $\mathbf{x}^2 = (z, \frac{3}{2}P_2^* - \frac{3}{2}p^* + z - \frac{3}{2}p_3 - \frac{1}{3})$, $\mathbf{x}^3 = (\frac{3}{4}p^* - \frac{3}{4}P_2^* - \frac{1}{2}y + \frac{3}{4}p_3 + \frac{7}{6}, y)$, $\mathbf{x}^4 = (\frac{3}{4}p^* - \frac{3}{4}P_2^* - \frac{1}{2}z + \frac{3}{4}p_3 + \frac{7}{6}, z)$.

[45] In Figure 32, $\mathbf{x}^1 = (\frac{3}{2}p^* - \frac{3}{2}P_2^* + \frac{3}{2}p_3 + \frac{1}{3}, 0)$, $\mathbf{x}^2 = (z, z - \frac{3}{2}p^* + \frac{3}{2}P_2^* - \frac{3}{2}p_3 - \frac{1}{3})$, $\mathbf{x}^3 = (1, \frac{3}{2}p^* - \frac{3}{2}P_2^* + \frac{3}{2}p_3 + \frac{1}{3})$, $\mathbf{x}^4 = (\frac{3}{4}p^* - \frac{3}{4}P_2^* - \frac{1}{2}z + \frac{3}{4}p_3 + \frac{7}{6}, z)$.

Proof that $p_3 = p^*$ is a best reply for firm 3 Since $\pi_3(p_3) = p_3 D_3(p_3)$, from (39) we obtain

$$\pi_3'(p_3) = \begin{cases} -\frac{27}{2}p_3^2 + (18P_2^* - 18p^* - 13)\,p_3 - \frac{9}{2}(P_2^*)^2 \\ \quad +9P_2^*p^* + 2P_2^* - \frac{9}{2}(p^*)^2 + \frac{5}{2}p^* + \frac{7}{9} & \text{if } 0 \le p_1 \le \frac{1}{5}p^* + \frac{2}{5}P_2^* + \frac{2}{45} \\ \frac{2943}{32}p_3^2 - (\frac{153}{4}P_2^* + \frac{369}{8}p^* + \frac{77}{4})p_3 + \frac{9}{8}(P_2^*)^2 \\ \quad +\frac{117}{8}P_2^*p^* + \frac{13}{4}P_2^* - \frac{99}{32}(p^*)^2 + \frac{25}{8}p^* + \frac{61}{72} & \text{if } \frac{1}{5}p^* + \frac{2}{5}P_2^* + \frac{2}{45} < p_1 \le \frac{1}{2}p^* + \frac{1}{4}P_2^* + \frac{1}{9} \\ \frac{1215}{32}p_3^2 - (\frac{81}{4}P_2^* + \frac{81}{8}p^* + \frac{45}{4})p_3 \\ \quad +\frac{9}{4}P_2^* + \frac{9}{8}p^* + \frac{81}{8}P_2^*p^* - \frac{243}{32}(p^*)^2 + \frac{5}{8} & \text{if } \frac{1}{2}p^* + \frac{1}{4}P_2^* + \frac{1}{9} < p_1 \le 2P_2^* - 3p^* + \frac{2}{9} \\ \frac{81}{2}p_3^2 - (27P_2^* + 12)\,p_3 + \frac{27}{8}(P_2^*)^2 + 3P_2^* + \frac{2}{3} & \text{if } 2P_2^* - 3p^* + \frac{2}{9} < p_1 \le \frac{1}{2}P_2^* + \frac{2}{9} \end{cases}$$

and now we prove that $\pi_3'(p_3) < 0$ for $p_3 \in (0, p^*)$, $\pi_3'(p_3) > 0$ for $p_3 \in (p^*, \frac{1}{2}P_2^* + \frac{2}{9})$.

For p_3 in the interval $[0, \frac{1}{5}p^* + \frac{2}{5}P_2^* + \frac{2}{45}]$, $\pi_3'(0) = 1.281 > 0$, $\pi_3'(p^*) = 0$ (see (37)), $\pi_3'(\frac{1}{5}p^* + \frac{2}{5}P_2^* + \frac{2}{45}) = -0.355$, and since π_3' is a concave quadratic function for p_3 in the interval, it is positive for $p_3 \in (0, p^*)$, it is negative for $p_3 \in (p^*, \frac{1}{5}p^* + \frac{2}{5}P_2^* + \frac{2}{45})$. For p_3 in the interval $(\frac{1}{5}p^* + \frac{2}{5}P_2^* + \frac{2}{45}, \frac{1}{2}p^* + \frac{1}{4}P_2^* + \frac{1}{9}]$ we have $\pi_3'(\frac{1}{5}p^* + \frac{2}{5}P_2^* + \frac{2}{45}) < 0$, and $\pi_3'(\frac{1}{2}p^* + \frac{1}{4}P_2^* + \frac{1}{9}) = -0.375$; since π_3' is convex in this interval, it follows that $\pi_3'(p) < 0$ for each p in the interval. For the intervals $(\frac{1}{2}p^* + \frac{1}{4}P_2^* + \frac{1}{9}, 2P_2^* - 3p^* + \frac{2}{9}]$ and $(2P_2^* - 3p^* + \frac{2}{9}, \frac{1}{2}P_2^* + \frac{2}{9}]$ we can apply the same argument as $\pi_3'(\frac{1}{2}p^* + \frac{1}{4}P_2^* + \frac{1}{9}) < 0$, $\pi_3'(2P_2^* - 3p^* + \frac{2}{9}) = -0.367$, $\pi_3'(\frac{1}{2}P_2^* + \frac{2}{9}) = 0$ and π_3' is convex both in $(\frac{1}{2}p^* + \frac{1}{4}P_2^* + \frac{1}{9}, 2P_2^* - 3p^* + \frac{2}{9}]$ and in $(2P_2^* - 3p^* + \frac{2}{9}, \frac{1}{2}P_2^* + \frac{2}{9}]$.

7.2 Proof of Lemma 5

Suppose that $\alpha \in (0, \frac{5}{9})$. Since $\pi_3(p_3) = p_3 D_3(p_3)$, from (15) we obtain that the first order condition for p_3, at $p_3 = p$, is $\frac{3}{2}p - \frac{3}{2}\alpha + \frac{3}{2}p_2 + \frac{1}{3} - 6p = 0$. Combining this with (14) yields p^*, p_2^* in (16). In order to see that the prices in (16) constitute a NE of $\gamma_\alpha^\varnothing$ when $\alpha \in (0, \frac{5}{9})$, notice that given $p_1 = p_3 = p^*$, we know from (14) that p_2^* is a best reply for firm 2. From the point of view of firm 3, given $p_1 = p^*$, $p_2 = p_2^*$ notice that the inequality $C_3(x_A) < \min\{C_1(x_A), C_2(x_A)\}$ is equivalent to $x_A \in (0, \frac{1}{6} - \frac{3}{10}\alpha - \frac{3}{2}p_3) \cup (\frac{1}{2} + \frac{9}{10}\alpha + \frac{3}{2}p_3, 1)$ if $p_3 \in (0, p^*)$, is equivalent to $x_A \in (\frac{1}{2} + \frac{9}{10}\alpha + \frac{3}{2}p_3, \frac{7}{6} - \frac{3}{10}\alpha - \frac{3}{2}p_3)$ if $p_3 \in (p^*, \frac{2}{9} - \frac{2}{5}\alpha)$, is violated for each $x \in [0, 1)$ if $p_3 \ge \frac{2}{9} - \frac{2}{5}\alpha$. Hence, for each $p_3 \in (0, \frac{2}{9} - \frac{2}{5}\alpha)$, $D_3(p_3)$ is equal to $\frac{2}{3} - 3p_3 - \frac{6}{5}\alpha$ and it is immediate that π_3 is maximized at $p_3 = p^*$.

Suppose that $\alpha \ge \frac{5}{9}$. In order to see that the prices in (17) constitute a NE of $\gamma_\alpha^\varnothing$, notice that each consumer buys only from firm 2, hence the profit of firm 3 is 0. Hence no $p_3 > 0$ is a profitable deviation for firm 3, as it leaves firm 3's demand equal to 0. For firm 2, (14) implies that p_2^* is a best reply for firm 2, given that $p = 0$, $\alpha \ge \frac{5}{9}$.

7.3 Proof of Lemma 6

In Subsection 5.2 we have established that given $p_1 = p_3 = p$, the best reply for firm 2 is given by $br_2(p, \gamma_\alpha^2)$ in (19). Here we consider the point of view of firm 3 (a similar argument applies for firm 1). Given that firm 1, firm 2 play $p_1 = p$, $P_2 = P$, we derive the demand function of firm 3 for $p_3 > p$, which allows to derive the first order condition from which (jointly with (19)) p^*, P_2^* in Lemma 6 are obtained. In the Supplementary Material we derive firm 3's complete demand function and show that $p_3 = p^*$ is a best reply for firm 3.

7.3.1 Demand function for firm 3 for $p_3 > p$

First we notice that the demand for firm 3 is 0 if $p_3 > p + \frac{2}{9}$, because then each consumer prefers product A_1 to A_3 and product B_1 to B_3. Given $p_3 \in (p, p + \frac{2}{9}]$, we solve the inequality $C_{ij}(x_A, x_B) < C_{22}(x_A, x_B)$ in region Q_{ij}, for $ij = 13, 31, 33$, where Q_{13}, Q_{31}, Q_{33} are described in (10), with $y = \frac{1}{2} + \frac{3}{4}(p_3 - p)$ and

$z = 2 - 2y$. Moreover, we distinguish between Q_{13}^W and Q_{13}^E, and Q_{31}^N and Q_{31}^S as described by Figure 33:

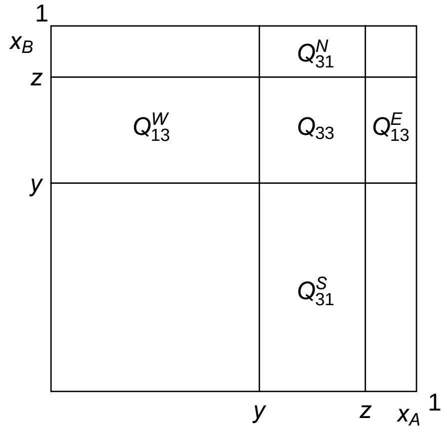


Figure 33
The sets $Q_{13}^W, Q_{13}^E, Q_{31}^N, Q_{31}^S,$:
Q_{33} in γ_α^2 when $p_3 > p$ (40)

Region Q_{33} In region Q_{33}, the inequality $C_{22}(x_A, x_B) > C_{33}(x_A, x_B)$ is equivalent to

$$x_B > 3\alpha - \frac{3}{2}P + 3p_3 - x_A + \frac{4}{3} \equiv f_1(x_A)$$

We use D_{33} to denote the area of the subset of Q_{33} that satisfies the inequality. We find that (i) if $f_1(y) < y$, that is if $p_3 < P - p - 2\alpha - \frac{2}{9}$, then $D_{33} = (z-y)^2$; (ii) if $y < f_1(y) < z$, that is if $P - p - 2\alpha - \frac{2}{9} < p_3 < \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45}$, then $D_{33} = (z-y)^2 - \frac{1}{2}\left(f_1(y) - y\right)^2$; (iii) if $f_1(z) < z < f_1(y)$, that is if $\frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} < p_3 < \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9}$, then $D_{33} = \frac{1}{2}(z - f_1(z))^2$; (iv) if $z < f_1(z)$, that is if $\frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9} < p_3$, then $D_{33} = 0$. Summarizing,

$$D_{33} = \begin{cases} (z-y)^2 & \text{if } p_3 < P - p - 2\alpha - \frac{2}{9} \text{ (i.e., if } f_1(y) < y) \\ (z-y)^2 - \frac{1}{2}\left(f_1(y) - y\right)^2 & \text{if } P - p - 2\alpha - \frac{2}{9} < p_3 < \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} \\ & \text{(i.e., if } y < f_1(y) < z) \\ \frac{1}{2}\left(z - f_1(z)\right)^2 & \text{if } \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} < p_3 < \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9} \\ & \text{(i.e., if } f_1(z) < z < f_1(y)) \\ 0 & \text{if } \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9} < p_3 \\ & \text{(i.e., if } z < f_1(z)) \end{cases} \tag{41}$$

Region Q_{13}^W In region Q_{13}^W, the inequality $C_{22}(x_A, x_B) > C_{13}(x_A, x_B)$ is equivalent to

$$x_B > \frac{3}{2}p - \frac{3}{2}P + 3\alpha + \frac{3}{2}p_3 + x_A + \frac{1}{3} \equiv f_2(x_A)$$

We use D_{13}^W to denote the area of the subset of Q_{13}^W that satisfies the inequality. We find that (i) if $f_2(y) < y$, that is if $p_3 < P - p - 2\alpha - \frac{2}{9}$ then $D_{13}^W = y(z-y)$; (ii) if $y < f_2(y) < z$, that is if $P - p - 2\alpha - \frac{2}{9} < p_3 < \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45}$, then $D_{13}^W = y(z-y) - \frac{1}{2}\left(f_2(y) - y\right)^2$; (iii) if $z < f_2(y)$ and $f_2(0) < y$, that is if $\frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} < p_3 < 2P - 3p - 4\alpha + \frac{2}{9}$, then $D_{13}^W = \left(y + z - \frac{1}{2}f_2(y) - \frac{1}{2}f_2(z)\right)(z-y)$; (iv) if $y < f_2(0) < z$, that is if $2P - 3p - 4\alpha + \frac{2}{9} < p_3 < \frac{1}{2}P - \alpha + \frac{2}{9}$, then $D_{13}^W = \frac{1}{2}(z - f_2(0))^2$; (v) if $z < f_2(0)$, that is if $\frac{1}{2}P - \alpha + \frac{2}{9} < p_3$, then $D_{13}^W = 0$. Summarizing,

$$D_{13}^W = \begin{cases} y(z-y) & \text{if } p_3 < P - p - 2\alpha - \frac{2}{9} \text{ (i.e., if } f_2(y) < y) \\ y(z-y) - \frac{1}{2}(f_2(y)-y)^2 & \text{if } P - p - 2\alpha - \frac{2}{9} < p_3 < \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} \\ & \text{(i.e., if } y < f_2(y) < z) \\ \left(y + z - \frac{1}{2}f_2(y) - \frac{1}{2}f_2(z)\right)(z-y) & \text{if } \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} < p_3 < 2P - 3p - 4\alpha + \frac{2}{9} \\ & \text{(i.e., if } z < f_2(y),\ f_2(0) < y) \\ \frac{1}{2}(z - f_2(0))^2 & \text{if } 2P - 3p - 4\alpha + \frac{2}{9} < p_3 < \frac{1}{2}P - \alpha + \frac{2}{9} \\ & \text{(i.e., if } y < f_2(0) < z) \\ 0 & \text{if } \frac{1}{2}P - \alpha + \frac{2}{9} < p_3 \text{ (i.e., if } z < f_2(0)) \end{cases} \tag{42}$$

Region Q_{13}^E In region Q_{13}^E, the inequality $C_{22}(x_A, x_B) > C_{13}(x_A, x_B)$ is equivalent to

$$x_B > \frac{3}{2}p - \frac{3}{2}P + 3\alpha + \frac{3}{2}p_3 - 2x_A + \frac{7}{3} \equiv f_3(x_A) \tag{43}$$

We use D_{13}^E to denote the area of the subset of Q_{13}^E that satisfies the inequality. We find that (i) if $f_3(z) < y$, that is if $p_3 < \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45}$, then $D_{13}^E = (1-z)(z-y)$; (ii) if $y < f_3(z)$ and $f_3(1) < y$ (this implies $f_3(z) < z$), that is if $\frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} < p_3 < 2P - 3p - 4\alpha + \frac{2}{9}$, then $D_{13}^E = (1-z)(z-y) - \frac{1}{4}\left(f_3(z) - y\right)^2$; (iii) if $f_3(z) < z$ and $y < f_3(1)$, that is if $2P - 3p - 4\alpha + \frac{2}{9} < p_3 < \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9}$, then $D_{13}^E = (z - \frac{1}{2}f_3(z) - \frac{1}{2}f_3(1))(1-z)$; (iv) if $z < f_3(z)$ and $f_3(1) < z$, that is if $\frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9} < p_3 < \frac{1}{2}P - \alpha + \frac{2}{9}$, then $D_{13}^E = \frac{1}{4}\left(z - f_3(1)\right)^2$; (v) if $z < f_3(1)$, that is if $\frac{1}{2}P - \alpha + \frac{2}{9} < p_3$, then $D_{13}^E = 0$. Summarizing,[46]

$$D_{13}^E = \begin{cases} (1-z)(z-y) & \text{if } p_3 < \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} \\ & \text{(i.e., if } f_3(z) < y) \\ (1-z)(z-y) - \frac{1}{4}\left(f_3(z) - y\right)^2 & \text{if } \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} < p_3 < 2P - 3p - 4\alpha + \frac{2}{9} \\ & \text{(i.e., if } y < f_3(z),\ f_3(1) < y) \\ (z - \frac{1}{2}f_3(z) - \frac{1}{2}f_3(1))(1-z) & \text{if } 2P - 3p - 4\alpha + \frac{2}{9} < p_3 < \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9} \\ & \text{(i.e., if } f_3(z) < z,\ y < f_3(1)) \\ \frac{1}{4}\left(z - f_3(1)\right)^2 & \text{if } \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9} < p_3 < \frac{1}{2}P - \alpha + \frac{2}{9} \\ & \text{(i.e., if } z < f_3(z),\ f_3(1) < z) \\ 0 & \text{if } \frac{1}{2}P - \alpha + \frac{2}{9} < p_3 \\ & \text{(i.e., if } z < f_3(1)) \end{cases} \tag{44}$$

Total demand We have neglected the regions Q_{31}^N and Q_{31}^S as $D_{31}^N = D_{13}^E$ and $D_{31}^S = D_{13}^W$. Therefore, the total demand for firm 3 is $D_3(p_3) = 2D_{33} + 2D_{13}^W + 2D_{13}^E$, and from (41), (42), (44) we obtain

$$D_3(p_3) = \begin{cases} 2z - 2y & \text{if } p_3 < P - p - 2\alpha - \frac{2}{9} \\ 2z - 2y - \left(f_1(y) - y\right)^2 - \left(f_2(y) - y\right)^2 & \text{if } P - p - 2\alpha - \frac{2}{9} < p_3 < \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} \\ \left(z - f_1(z)\right)^2 + \left(2y + 2z - f_2(y) - f_2(z)\right)(z-y) & \\ \quad +2(1-z)(z-y) - \frac{1}{2}\left(f_3(z) - y\right)^2 & \text{if } \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} < p_3 < 2P - 3p - 4\alpha + \frac{2}{9} \\ \left(z - f_1(z)\right)^2 + \left(z - f_2(0)\right)^2 & \\ \quad +(2z - f_3(z) - f_3(1))(1-z) & \text{if } 2P - 3p - 4\alpha + \frac{2}{9} < p_3 < \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9} \\ \left(z - f_2(0)\right)^2 + \frac{1}{2}\left(z - f_3(1)\right)^2 & \text{if } \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9} < p_3 < \frac{1}{2}P - \alpha + \frac{2}{9} \\ 0 & \text{if } \frac{1}{2}P - \alpha + \frac{2}{9} < p_3 \end{cases} \tag{45}$$

[46] In fact, D_{13}^E in (44) applies as long as $p \geq \frac{1}{2}P - \alpha + \frac{2}{63}$, which is equivalent to $2P - 3p - 4\alpha + \frac{2}{9} \leq \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9}$. Although the opposite inequality holds in NE for α close to zero, in these cases the expression of D_{13}^E does not depend on whether $p \geq \frac{1}{2}P - \alpha + \frac{2}{63}$ holds or not, as long as p_3 is close to p. We provide a complete analysis in the Supplementary Material.

7.3.2 The Candidate Equilibrium

The case of $\alpha \le \frac{13}{180}$ Consider p, P such that $P - p - 2\alpha - \frac{2}{9} \le p \le \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45}$, and $p + \alpha < \frac{5}{36}$; these inequalities hold for the NE when $\alpha = 0$: see Lemma 3. Then, for p_3 slightly larger than p we find, from (45), $D_3(p_3) = 2z - 2y - (f_1(y) - y)^2 - (f_2(y) - y)^2$, which is the expression in (20). Hence

$$\pi_3'(p_3) = -\frac{27}{2}p_3^2 + (18P - 18p - 36\alpha - 13)p_3 + \frac{9}{2}p + 1 - \frac{9}{2}(p + 2\alpha - P + \frac{2}{9})^2 \tag{46}$$

and the first order condition for p_3, at $p_3 = p$, is $-\frac{9}{2}P^2 + 27Pp + 18P\alpha + 2P - 36p^2 - 54p\alpha - \frac{21}{2}p - 18\alpha^2 - 4\alpha + \frac{7}{9} = 0$. Jointly with $P = \frac{2}{3}p + \frac{2}{3}\alpha + \frac{2}{27}$ from (19), this yields p^*, P_2^* in (22), which satisfy the inequalities $P - p - 2\alpha - \frac{2}{9} \le p \le \frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45}$ and $p + \alpha \le \frac{5}{36}$ for each $\alpha \le \frac{13}{180}$, but violate the latter two if $\alpha > \frac{13}{180}$.

The case of $\alpha \in (\frac{13}{180}, \frac{5}{9})$ For $\alpha > \frac{13}{180}$, we know that in NE the inequality $\frac{2}{5}P + \frac{1}{5}p - \frac{4}{5}\alpha + \frac{2}{45} < p$ holds, which is equivalent to $p > \frac{1}{2}P - \alpha + \frac{1}{18}$. However, with reference to (45), $p < 2P - 3p - 4\alpha + \frac{2}{9}$ cannot hold as it is equivalent to $p < \frac{1}{2}P - \alpha + \frac{1}{18}$. Then we consider p, P such that $p + \alpha > \frac{5}{36}$ and $2P - 3p - 4\alpha + \frac{2}{9} < p < \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9}$ and for p_3 slightly larger than p, we have $D_3(p_3) = (z - f_1(z))^2 + (z - f_2(0))^2 + (2z - f_3(z) - f_3(1))(1 - z)$, which is the expression in (21). Then

$$\pi_3'(p_3) = \frac{189}{2}p_3^2 + (90\alpha - 36p - 45P - 20)\,p_3 + \frac{9}{2}P^2 + \frac{9}{2}Pp - 18P\alpha + 4P + \frac{9}{2}p^2 - 9p\alpha + 2p + 18\alpha^2 - 8\alpha + \frac{8}{9} \tag{47}$$

and the first order condition with respect to p_3, at $p_3 = p$, is $\frac{9}{2}P^2 - \frac{81}{2}Pp - 18P\alpha + 4P + 63p^2 + 81p\alpha - 18p + 18\alpha^2 - 8\alpha + \frac{8}{9} = 0$. Jointly with $P = \frac{1}{27}\sqrt{324(p+\alpha)^2 - 144(p+\alpha) + 70} + \frac{4}{3}(p+\alpha) - \frac{8}{27}$ from (19), this yields p^*, P_2^* in (23), which satisfy $2P - 3p - 4\alpha + \frac{2}{9} < \frac{1}{4}P + \frac{1}{2}p - \frac{1}{2}\alpha + \frac{1}{9}$ and $p + \alpha > \frac{5}{36}$ for each $\alpha \in (\frac{13}{180}, \frac{5}{9})$.

7.4 Proof of Lemma 7

For each $\alpha \in (0, \frac{5}{9})$, we consider NE of γ_α^3, (p_1^*, p_2^*, P_3^*), such that $p_2^* - \alpha < p_1^*$, $p_1^* < p_2^* - \alpha + \frac{2}{9}$, $\frac{1}{2}P_3^* < p_2^* - \alpha + \frac{2}{9}$. While the first inequality is intuitive as $\alpha > 0$, the second and third inequality are necessary for firm 1 and firm 3 to have positive demand. For instance, if $p_1^* \ge p_2^* - \alpha + \frac{2}{9}$ then each consumer prefers product A (B) of firm 2 to product A (B) of firm 1, hence buys no product of firm 1. If there exists a NE such that $p_1^* \ge p_2^* - \alpha + \frac{2}{9}$, then also firm 3 has zero demand (that is $\frac{1}{2}P_3^* \ge p_2^* - \alpha + \frac{2}{9}$), or else firm 1 can profitably deviate by setting $p_1 = \frac{1}{2}P_3^*$. Moreover, it is necessary that $p_2^* - \alpha + \frac{2}{9} = 0$,[47] and $p_1^* \ge 0$, $P_3^* \ge 0$ with at least one equality. But then it is profitable for firm 2 to slightly increase p_2 because firm 2's demand has zero elasticity at the equilibrium, such that a small price increase makes firm 2 lose a very small amount of customers but increases the firm's revenue from inframarginal consumers.

Given p_1, p_2 such that $p_2 - \alpha < p_1 < p_2 - \alpha + \frac{2}{9}$, we examine how a consumer located at $x \in [0, 1)$ for a product (which could be A or B) chooses between the product of firm 1 and the product of firm 2 if there is no firm 3:

$$\begin{cases} \text{for} \quad x \in [0, \frac{2}{3}), & C_2(x) < C_1(x) \Longleftrightarrow x > y \equiv \frac{1}{3} - \frac{3}{2}(p_1 - p_2 + \alpha) \\ \text{for} \quad x \in [\frac{2}{3}, 1), & C_2(x) < C_1(x) \Longleftrightarrow x < z \equiv \frac{5}{6} + \frac{3}{4}(p_1 - p_2 + \alpha) \end{cases} \tag{48}$$

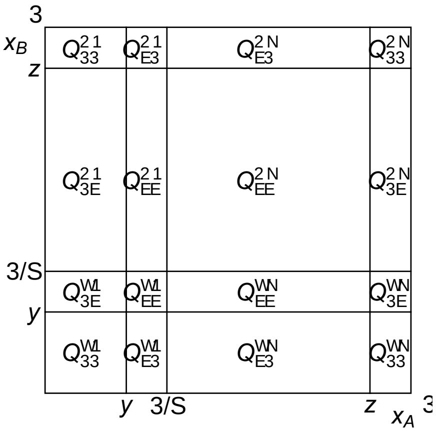
and notice that $y \in (0, \frac{1}{3})$, $z \in (\frac{5}{6}, 1)$ since $0 < p_1 - p_2 + \alpha < \frac{2}{9}$. As a consequence, we partition $[0, 1) \times [0, 1)$

[47] If $p_2^* - \alpha + \frac{2}{9} < 0$, then firm 2 may increase p_2 and still sell both products to all consumers. If $p_2^* - \alpha + \frac{2}{9} > 0$, then setting $p_1 > 0$ smaller than $p_2^* - \alpha + \frac{2}{9}$ is a profitable deviation for firm 1 (for firm 3 it is profitable to set P_3 smaller than $2p_2^* - 2\alpha + \frac{4}{9}$).

into the following subsets:

$$\begin{cases} Q_{11}^{SW} = [0,y) \times [0,y), & Q_{21}^{SW} = [y,\frac{1}{3}) \times [0,y), & Q_{21}^{SE} = [\frac{1}{3},z) \times [0,y), & Q_{11}^{SE} = [z,1) \times [0,y), \\ Q_{12}^{SW} = [0,y) \times [y,\frac{1}{3}), & Q_{22}^{SW} = [y,\frac{1}{3}) \times [y,\frac{1}{3}), & Q_{22}^{SE} = [\frac{1}{3},z) \times [y,\frac{1}{3}), & Q_{12}^{SE} = [z,1) \times [y,\frac{1}{3}), \\ Q_{12}^{NW} = [0,y) \times [\frac{1}{3},z), & Q_{22}^{NW} = [y,\frac{1}{3}) \times [\frac{1}{3},z), & Q_{22}^{NE} = [\frac{1}{3},z) \times [\frac{1}{3},z), & Q_{12}^{NE} = [z,1) \times [\frac{1}{3},z), \\ Q_{11}^{NW} = [0,y) \times [z,1), & Q_{21}^{NW} = [y,\frac{1}{3}) \times [z,1), & Q_{21}^{NE} = [\frac{1}{3},z) \times [z,1), & Q_{11}^{NE} = [z,1) \times [z,1) \end{cases} \tag{49}$$

Figure 34
The partition of $[0,1) \times [0,1)$
described in (49)



In order to derive the demand function for firm 1, with reference to the regions in (49) we need to determine the area of the (x_A, x_B) that satisfy the inequality $C_{11}(x_A, x_B) < C_{33}(x_A, x_B)$ in the regions Q_{11}^{SW}, Q_{11}^{SE}, Q_{11}^{NE}, Q_{11}^{NW}. Likewise, we need to determine the area of the (x_A, x_B) that satisfy the inequality $C_{21}(x_A, x_B) < C_{33}(x_A, x_B)$ in the regions Q_{21}^{SW}, Q_{21}^{SE}, Q_{21}^{NE}, Q_{21}^{NW}, the area of the (x_A, x_B) that satisfy the inequality $C_{12}(x_A, x_B) < C_{33}(x_A, x_B)$ in the regions Q_{12}^{SW}, Q_{12}^{SE}, Q_{12}^{NE}, Q_{12}^{NW}.[48] This is done in the Supplementary Material in which, for p_1 between $\max\{p_2 - \alpha, \frac{1}{2}P_3\}$ and $\frac{2}{5}P_3 + \frac{1}{5}(p_2 - \alpha) + \frac{2}{45}$, we find $D_1(p_1)$ as in (24); for p_1 between $\frac{2}{5}P_3 + \frac{1}{5}(p_2 - \alpha) + \frac{2}{45}$ and $\frac{1}{3}P_3 + \frac{1}{3}(p_2 - \alpha) + \frac{2}{27}$ we find $D_1(p_1)$ as in (25).

In order to derive the demand function for firm 2, with reference to the regions in (49) we solve the inequality $C_{22}(x_A, x_B) < C_{33}(x_A, x_B)$ in the regions Q_{22}^{SW}, Q_{22}^{SE}, Q_{22}^{NE}, Q_{22}^{NW}, the inequality $C_{21}(x_A, x_B) < C_{33}(x_A, x_B)$ in the regions Q_{21}^{SW}, Q_{21}^{SE}, Q_{21}^{NE}, Q_{21}^{NW}, the inequality $C_{12}(x_A, x_B) < C_{33}(x_A, x_B)$ in the regions Q_{12}^{SW}, Q_{12}^{SE}, Q_{12}^{NE}, Q_{12}^{NW}. This is done in the Supplementary Material in which, for p_2 between $\alpha - 2P_3 + 5p_1 - \frac{2}{9}$ and $\alpha + p_1$, we find $D_2(p_2)$ as in (24); for p_2 between $\alpha - P_3 + 3p_1 - \frac{2}{9}$ and $\alpha - 2P_3 + 5p_1 - \frac{2}{9}$ we find $D_2(p_2)$ as in (25).

In order to derive the demand function for firm 3, with reference to the regions in (49) we solve the inequality $C_{33}(x_A, x_B) < C_{11}(x_A, x_B)$ in the regions Q_{11}^{SW}, Q_{11}^{SE}, Q_{11}^{NE}, Q_{11}^{NW}, the inequality $C_{33}(x_A, x_B) < C_{21}(x_A, x_B)$ in the regions Q_{21}^{SW}, Q_{21}^{SE}, Q_{21}^{NE}, Q_{21}^{NW}, the inequality $C_{33}(x_A, x_B) < C_{22}(x_A, x_B)$ in the regions Q_{22}^{SW}, Q_{22}^{SE}, Q_{22}^{NE}, Q_{22}^{NW}, the inequality $C_{33}(x_A, x_B) < C_{12}(x_A, x_B)$ in the regions Q_{12}^{SW}, Q_{12}^{SE}, Q_{12}^{NE}, Q_{12}^{NW}. This is done in the Supplementary Material, in which for P_3 between $\frac{5}{2}p_1 - \frac{1}{2}(p_2 - \alpha) - \frac{1}{9}$ and $p_1 + p_2 - \alpha + \frac{2}{9}$, we find $D_3(P_3)$ as in (24); for P_3 between $\max\{4p_1 - 2(p_2 - \alpha) - \frac{4}{9}, \frac{1}{2}p_1 + \frac{3}{2}(p_2 - \alpha) - \frac{1}{9}\}$ and $\frac{5}{2}p_1 - \frac{1}{2}(p_2 - \alpha) - \frac{1}{9}$ we find $D_3(P_3)$ as in (25).

Summarizing, (24) applies if

$$\begin{cases} \max\{p_2 - \alpha, \frac{1}{2}P_3\} < p_1 \le \frac{2}{5}P_3 + \frac{1}{5}(p_2 - \alpha) + \frac{2}{45}, \qquad \alpha - 2P_3 + 5p_1 - \frac{2}{9} \le p_2 < \alpha + p_1 \\ \frac{5}{2}p_1 - \frac{1}{2}(p_2 - \alpha) - \frac{1}{9} \le P_3 < p_1 + p_2 - \alpha + \frac{2}{9} \end{cases} \tag{50}$$

[48] In fact, we can neglect Q_{11}^{NW}, as in Q_{11}^{NW} the set of (x_A, x_B) that satisfy the inequality $C_{11}(x_A, x_B) < C_{33}(x_A, x_B)$ has the same area as the set of (x_A, x_B) that satisfy $C_{11}(x_A, x_B) < C_{33}(x_A, x_B)$ in Q_{11}^{SE}. A similar remark explains why we can neglect Q_{12}^{SW}, Q_{12}^{SE}, Q_{12}^{NE}, Q_{12}^{NW}, and also applies to the derivation of the demand functions of firms 2 and 3.

and from (24) we obtain the following first order conditions for p_1, p_2, P_3:[49]

$$\begin{cases} (18P_3 - 18(p_2-\alpha) - 13)\,p_1 - \frac{27}{2}p_1^2 - \frac{9}{2}P_3^2 \\ +9P_3(p_2-\alpha) + 2P_3 - \frac{9}{2}(p_2-\alpha)^2 + \frac{5}{2}(p_2-\alpha) + \frac{7}{9} = 0 \\ -\frac{27}{2}p_2^2 + (18\alpha + 18P_3 - 18p_1 - 13)\,p_2 - \frac{9}{2}\alpha^2 - 9\alpha P_3 \\ +9\alpha p_1 + \frac{13}{2}\alpha - \frac{9}{2}P_3^2 + 9P_3p_1 + 2P_3 - \frac{9}{2}p_1^2 + \frac{5}{2}p_1 + \frac{7}{9} = 0 \\ P_3 - (\frac{1}{3}p_1 - \frac{1}{3}\alpha + \frac{1}{3}p_2 + \frac{2}{27}) = 0 \end{cases} \tag{51}$$

By solving (51) numerically, we obtain a solution that satisfies (50) for $\alpha \in (0, \frac{26}{77}]$ but not for $\alpha > \frac{26}{77}$. In particular, for $\alpha > \frac{26}{77}$ the inequality $p_1 \le \frac{2}{5}P_3 + \frac{1}{5}(p_2-\alpha) + \frac{2}{45}$ is violated, therefore also $\alpha - 2P_3 + 5p_1 - \frac{2}{9} \le p_2$ and $\frac{5}{2}p_1 - \frac{1}{2}(p_2-\alpha) - \frac{1}{9} \le P_3$ are violated. However, (25) applies if

$$\begin{cases} \frac{2}{5}P_3 + \frac{1}{5}(p_2-\alpha) + \frac{2}{45} < p_1 < \frac{1}{3}P_3 + \frac{1}{3}(p_2-\alpha) + \frac{2}{27}, \qquad \alpha - P_3 + 3p_1 - \frac{2}{9} < p_2 < \alpha - 2P_3 + 5p_1 - \frac{2}{9} \\ \max\{4p_1 - 2(p_2-\alpha) - \frac{4}{9}, \frac{1}{2}p_1 + \frac{3}{2}(p_2-\alpha) - \frac{1}{9}\} < P_3 < \frac{5}{2}p_1 - \frac{1}{2}(p_2-\alpha) - \frac{1}{9} \end{cases} \tag{52}$$

and (25) yields the following first order conditions for p_1, p_2, P_3:

$$\begin{cases} \frac{2943}{32}p_1^2 - \left(\frac{153}{4}P_3 + \frac{369}{8}(p_2-\alpha) + \frac{77}{4}\right)p_1 + \frac{9}{8}P_3^2 + \frac{13}{4}P_3 \\ +\frac{117}{8}P_3(p_2-\alpha) - \frac{99}{32}(p_2-\alpha)^2 + \frac{25}{8}(p_2-\alpha) + \frac{61}{72} = 0 \\ -\frac{459}{32}p_2^2 + \left(\frac{153}{8}\alpha + \frac{63}{4}P_3 - \frac{99}{8}p_1 - \frac{53}{4}\right)p_2 - \frac{153}{32}\alpha^2 - \frac{63}{8}\alpha P_3 + \frac{99}{16}\alpha p_1 \\ +\frac{53}{8}\alpha - \frac{45}{8}P_3^2 + \frac{117}{8}P_3p_1 + \frac{7}{4}P_3 - \frac{369}{32}p_1^2 + \frac{25}{8}p_1 + \frac{55}{72} = 0 \\ \frac{27}{4}P_3^2 + \left(\frac{9}{2}p_1 - \frac{45}{2}(p_2-\alpha) - 5\right)P_3 - \frac{153}{16}p_1^2 + \frac{13}{4}p_1 \\ +\frac{117}{8}p_1(p_2-\alpha) + \frac{63}{16}(p_2-\alpha)^2 + \frac{7}{4}(p_2-\alpha) + \frac{7}{36} = 0 \end{cases} \tag{53}$$

By solving (53) numerically, we obtain a solution that satisfies (52) for each $\alpha \in (\frac{26}{77}, \frac{5}{9})$. In the Supplementary Material we show that the solution we obtain is a NE for each $\alpha \in (0, \frac{5}{9})$.

7.5 Proof for Lemma 8

Here we provide some details about the derivation of (27)-(28), (30)-(31), and the derivation of the prices P^*, P_2^* in Lemma 8. In the Supplementary Material we provide full details and prove that Lemma 8 identifies a NE of γ_α^{123} for each $\alpha \in (0, \frac{5}{9})$.

7.5.1 Demand function for firm 2

Given $P_1 = P_3 = P$ and $\delta = \frac{1}{2}P_2 - \frac{1}{2}P - \alpha$, in order to determine the set of (x_A, x_B) which satisfy (26) we partition $[0,1) \times [0,1)$ into the following nine regions, represented in Figure 35:

$$\begin{array}{ccc} Q^{SW} = [0,\frac{1}{3}) \times [0,\frac{1}{3}), & Q^{SM} = [\frac{1}{3},\frac{2}{3}) \times [0,\frac{1}{3}), & Q^{SE} = [\frac{2}{3},1) \times [0,\frac{1}{3}), \\ Q^{MW} = [0,\frac{1}{3}) \times [\frac{1}{3},\frac{2}{3}), & Q^{MM} = [\frac{1}{3},\frac{2}{3}) \times [\frac{1}{3},\frac{2}{3}), & Q^{ME} = [\frac{2}{3},1) \times [\frac{1}{3},\frac{2}{3}), \\ Q^{NW} = [0,\frac{1}{3}) \times [\frac{2}{3},1), & Q^{NM} = [\frac{1}{3},\frac{2}{3}) \times [\frac{2}{3},1), & Q^{NE} = [\frac{2}{3},1) \times [\frac{2}{3},1). \end{array} \tag{54}$$

[49] The first order condition for P_3 is written taking into account that the derivative of the profit function of firm 3, given D_3 in (24), factors into $\frac{27}{2}\left(P_3 - (\frac{1}{3}p_1 - \frac{1}{3}\alpha + \frac{1}{3}p_2 + \frac{2}{27})\right)\left(P_3 - (p_1 - \alpha + p_2 + \frac{2}{9})\right)$, and since $\frac{1}{3}p_1 - \frac{1}{3}\alpha + \frac{1}{3}p_2 + \frac{2}{27} < p_1 - \alpha + p_2 + \frac{2}{9}$ (because $p_1 + \alpha - \frac{2}{9} < p_2$ in equilibrium, otherwise firm 1 has zero demand), it follows that $P_3 = p_1 - \alpha + p_2 + \frac{2}{9}$ is a minimum point for the profit of firm 3, $P_3 = \frac{1}{3}p_1 - \frac{1}{3}\alpha + \frac{1}{3}p_2 + \frac{2}{27}$ is a maximum point.

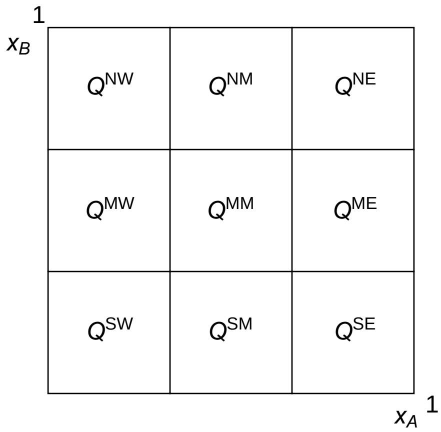


Figure 35:
the partition of $[0,1)\times[0,1)$
described in (54)

This partition is useful because

$$C_{22}(x_A,x_B) < C_{33}(x_A,x_B) \Leftrightarrow \begin{cases} x_B > \frac{1}{3} - x_A + \frac{3}{2}\delta & \text{if } (x_A,x_B)\in Q^{SW} \\ x_B > \frac{1}{2}x_A - \frac{1}{6} + \frac{3}{2}\delta & \text{if } (x_A,x_B)\in Q^{SM}\cup Q^{SE} \\ x_B < \frac{1}{3} + 2x_A - 3\delta & \text{if } (x_A,x_B)\in Q^{MW}\cup Q^{NW} \\ x_B < \frac{4}{3} - x_A - 3\delta & \text{if } (x_A,x_B)\in Q^{MM}\cup Q^{ME}\cup Q^{NM}\cup Q^{NE} \end{cases} \tag{55}$$

$$C_{22}(x_A,x_B) < C_{11}(x_A,x_B) \Leftrightarrow \begin{cases} x_B > \frac{2}{3} - x_A + 3\delta & \text{if } (x_A,x_B)\in Q^{SW}\cup Q^{SM}\cup Q^{MW}\cup Q^{MM} \\ x_B > 2x_A - \frac{4}{3} + 3\delta & \text{if } (x_A,x_B)\in Q^{SE}\cup Q^{ME} \\ x_B < \frac{2}{3} + \frac{1}{2}x_A - \frac{3}{2}\delta & \text{if } (x_A,x_B)\in Q^{NW}\cup Q^{NM} \\ x_B < \frac{5}{3} - x_A - \frac{3}{2}\delta & \text{if } (x_A,x_B)\in Q^{NE} \end{cases} \tag{56}$$

From (55)-(56) it is immediate to derive $D_2(P_2)=1$ if $\delta<-\frac{2}{9}$, $D_2(P_2)=0$ if $\delta\geq\frac{1}{9}$.

For $\delta\in[-\frac{2}{9},-\frac{1}{18})$, (55)-(56) identify the decagon in Figure 20.[50] Using $\mathbf{x}^6=(0,0)$, $\mathbf{x}^7=(1,0)$, $\mathbf{x}^8=(1,1)$, we see that the area of the decagon is equal to 1 minus $\mathcal{A}(\bar{\mathbf{x}}^1\mathbf{x}^6\mathbf{x}^1)+\mathcal{A}(\mathbf{x}^2\mathbf{x}^7\mathbf{x}^4\mathbf{x}^3)+\mathcal{A}(\mathbf{x}^5\mathbf{x}^8\bar{\mathbf{x}}^5)+\mathcal{A}(\bar{\mathbf{x}}^2\bar{\mathbf{x}}^3\bar{\mathbf{x}}^4\bar{\mathbf{x}}^7)$, in which $\mathcal{A}(\mathbf{z}^1...\mathbf{z}^k)$ is the area of the polygon with vertices $\mathbf{z}^1...\mathbf{z}^k$. Since $\mathcal{A}(\bar{\mathbf{x}}^1\mathbf{x}^6\mathbf{x}^1)+\mathcal{A}(\mathbf{x}^2\mathbf{x}^7\mathbf{x}^4\mathbf{x}^3)+\mathcal{A}(\mathbf{x}^5\mathbf{x}^8\bar{\mathbf{x}}^5)+\mathcal{A}(\bar{\mathbf{x}}^2\bar{\mathbf{x}}^3\bar{\mathbf{x}}^4\bar{\mathbf{x}}^7)=(\frac{2}{3}+3\delta)^2+2\left(\left(\frac{2}{3}+3\delta+\frac{2}{9}+\delta\right)(\frac{2}{9}+\delta)\frac{1}{2}+\frac{1}{2}(\frac{4}{9}+2\delta)(\frac{2}{9}+\delta)\right)=15(\frac{2}{9}+\delta)^2$, it follows that $D_2(P_2)=1-15(\frac{2}{9}+\delta)^2$.

If $\delta\in[-\frac{1}{18},\frac{1}{9})$, then (55)-(56) identifies the hexagon in Figure 21. We use $\mathcal{L}(\mathbf{z}^1\mathbf{z}^2)$ to denote the length of the segment that connects point $\mathbf{z}^1$ to point $\mathbf{z}^2$, and find that $\mathcal{L}(\mathbf{y}^1\bar{\mathbf{y}}^1)=\sqrt{2}(\frac{4}{9}-\delta)$, $\mathcal{L}(\mathbf{y}^2\bar{\mathbf{y}}^2)=\sqrt{2}(\frac{5}{9}-2\delta)$, $\mathcal{L}(\mathbf{y}^1\mathbf{y}^3)=3\sqrt{2}(\frac{1}{9}-\delta)$, hence $\mathcal{A}(\mathbf{y}^1\mathbf{y}^2...\bar{\mathbf{y}}^1)=\left(\sqrt{2}(\frac{4}{9}-\delta)+\sqrt{2}(\frac{5}{9}-2\delta)\right)\frac{1}{2}\cdot3\sqrt{2}(\frac{1}{9}-\delta)=3(1-3\delta)(\frac{1}{9}-\delta)$.

From $\delta=\frac{1}{2}P_2-\frac{1}{2}P-\alpha$ we obtain (27), and from $\Pi_2(P_2)=P_2D_2(P_2)$ we obtain $\Pi_2'(P_2)$ below, which yields (28):

$$\Pi_2'(P_2)=\begin{cases} -\frac{45}{4}P_2^2+(15P+30\alpha-\frac{20}{3})P_2+1-15(\frac{1}{2}P+\alpha-\frac{2}{9})^2 & \text{if } P+2\alpha-\frac{4}{9}\leq P_2<P+2\alpha-\frac{1}{9} \\ \frac{27}{4}P_2^2-(9P+18\alpha+4)P_2+\frac{1}{12}(3P+6\alpha+2)(9P+18\alpha+2) & \text{if } P+2\alpha-\frac{1}{9}\leq P_2<P+2\alpha+\frac{2}{9} \end{cases} \tag{57}$$

[50] Precisely, if $\delta\in[-\frac{2}{9},-\frac{1}{9})$ then (55)-(56) is satisfied for each $(x_A,x_B)\in Q^{SM}\cup Q^{MW}\cup Q^{MM}\cup Q^{ME}\cup Q^{NM}$ and the points $\mathbf{x}^1,...,\mathbf{x}^5,\bar{\mathbf{x}}^1,...,\bar{\mathbf{x}}^5$ all lie in $Q^{SW}\cup Q^{SE}\cup Q^{NE}\cup Q^{NW}$, whereas if $\delta\in(-\frac{1}{9},-\frac{1}{18}]$, then $\mathbf{x}^1,\mathbf{x}^2,\mathbf{x}^4,\mathbf{x}^5,\bar{\mathbf{x}}^5,\bar{\mathbf{x}}^4,\bar{\mathbf{x}}^2,\bar{\mathbf{x}}^1$ belong to $Q^{SM}\cup Q^{ME}\cup Q^{NM}\cup Q^{MW}$. But in both cases the area of the decagon is evaluated as described in the following.

7.5.2 Demand function of firm 3 and candidate equilibrium

Given $P_1 = P$ and $\mu = P_3 - P$, $\theta = P_3 - P_2 + 2\alpha$, in order to determine the set of (x_A, x_B) which satisfy (29), we notice that

$$C_{33}(x_A, x_B) < C_{22}(x_A, x_B) \Leftrightarrow \begin{cases} x_B \leq \frac{1}{3} - x_A - \frac{3}{4}\theta & \text{if } (x_A, x_B) \in Q^{SW} \\ x_B \leq \frac{1}{2}x_A - \frac{1}{6} - \frac{3}{4}\theta & \text{if } (x_A, x_B) \in Q^{SM} \cup Q^{SE} \\ x_B \geq \frac{1}{3} + 2x_A + \frac{3}{2}\theta & \text{if } (x_A, x_B) \in Q^{MW} \cup Q^{NW} \\ x_B \geq \frac{4}{3} - x_A + \frac{3}{2}\theta & \text{if } (x_A, x_B) \in Q^{MM} \cup Q^{ME} \cup Q^{NM} \cup Q^{NE} \end{cases} \tag{58}$$

$$C_{33}(x_A, x_B) < C_{11}(x_A, x_B) \Leftrightarrow \begin{cases} x_B \leq -\frac{3}{2}\mu - x_A & \text{if } (x_A, x_B) \in Q^{SW} \\ x_B \leq 2x_A - \frac{3}{2}\mu - 1 & \text{if } (x_A, x_B) \in Q^{SM} \\ x_B \leq 1 - x_A - \frac{3}{2}\mu & \text{if } (x_A, x_B) \in Q^{SE} \\ x_B \geq \frac{3}{4}\mu + \frac{1}{2}x_A + \frac{1}{2} & \text{if } (x_A, x_B) \in Q^{MW} \\ x_B \geq \frac{3}{4}\mu - x_A + 1 & \text{if } (x_A, x_B) \in Q^{MM} \\ x_B \geq \frac{3}{4}\mu + \frac{1}{2}x_A & \text{if } (x_A, x_B) \in Q^{ME} \\ x_B \leq 1 - x_A - \frac{3}{2}\mu & \text{if } (x_A, x_B) \in Q^{NW} \\ x_B \leq 2x_A - \frac{3}{2}\mu & \text{if } (x_A, x_B) \in Q^{NM} \\ x_B \leq 2 - x_A - \frac{3}{2}\mu & \text{if } (x_A, x_B) \in Q^{NE} \end{cases} \tag{59}$$

When α is close to 0, in equilibrium both μ and θ are close to 0. In this case the set of (x_A, x_B) which satisfy (58)-(59) is described by Figure 22 and its area is $\mathcal{A}(\mathbf{x}^1\mathbf{x}^2\mathbf{x}^3\mathbf{x}^4) + \mathcal{A}(\bar{\mathbf{x}}^1\bar{\mathbf{x}}^2\bar{\mathbf{x}}^3\bar{\mathbf{x}}^4) + \mathcal{A}(\mathbf{x}^5\mathbf{x}^6\mathbf{x}^7\bar{\mathbf{x}}^7\bar{\mathbf{x}}^6\bar{\mathbf{x}}^5)$, which reduces to (30). Precisely, $D_3(P_3)$ is given by (30) as long as P_3 is between $\frac{1}{2}P - \alpha + \frac{1}{2}P_2 - \frac{1}{9}$ and $2P_2 - 4\alpha - P + \frac{2}{9}$. Hence $\Pi_3'(P_3) = \frac{27}{4}P_3^2 + \left(9\alpha - \frac{9}{2}P - \frac{9}{2}P_2 - 4\right)P_3 + \frac{9}{2}PP_2 - 9P\alpha - \frac{9}{8}P^2 + P - \frac{9}{2}\alpha^2 + \frac{9}{2}\alpha P_2 - 2\alpha - \frac{9}{8}P_2^2 + P_2 + \frac{1}{3}$ and the first order condition for P_3, at $P_3 = P$, is $\frac{9}{8}P^2 - 3P - \frac{9}{2}\alpha^2 + \frac{9}{2}\alpha P_2 - 2\alpha - \frac{9}{8}P_2^2 + P_2 + \frac{1}{3} = 0$. Combining this with (28) (for the case of $P + 2\alpha \leq \frac{31}{90}$) yields

$$P^* = \frac{1}{13 - 54\alpha}\left(108\alpha^2 + 108(P_2^*)^2 - 270\alpha P_2^* - 23\alpha - \frac{727}{18} - 89\rho(\alpha) + \frac{89(18\alpha - 972\alpha^2 - 173)}{2916\rho(\alpha)}\right) \tag{60}$$

$$P_2^* = \alpha + \frac{11}{18} + \rho(\alpha) + \frac{972\alpha^2 - 18\alpha + 173}{2916\rho(\alpha)} \tag{61}$$

in which $\rho(\alpha) = \frac{1}{18}\sqrt[3]{162\alpha^2 - 453\alpha - \frac{76}{3} + (13 - 54\alpha)\sqrt{-432\alpha^4 - 184\alpha^3 - \frac{857}{3}\alpha^2 - \frac{506}{3}\alpha - \frac{27869}{729}}}$.[51] The prices in (60)-(61) satisfy $\frac{1}{2}P - \alpha + \frac{1}{2}P_2 - \frac{1}{9} \leq P \leq 2P_2 - 4\alpha - P + \frac{2}{9}$ for $\alpha \in (0, \frac{71}{630}]$, but violate the second inequality if $\alpha > \frac{71}{630}$.

Then, considering P_3 larger than $2P_2 - 4\alpha - P + \frac{2}{9}$, $D_3(P_3)$ is the area of the disconnected set in Figure 23, that is $\mathcal{A}(\mathbf{y}^1\mathbf{y}^2\mathbf{y}^3) + \mathcal{A}(\bar{\mathbf{y}}^1\bar{\mathbf{y}}^2\bar{\mathbf{y}}^3) + \mathcal{A}(\mathbf{y}^4\mathbf{y}^5\bar{\mathbf{y}}^5\bar{\mathbf{y}}^4)$, which reduces to (31). Hence $\Pi_3'(P_3) = 9P_3^2 + \left(\frac{21}{2}\alpha - \frac{3}{4}P - \frac{21}{4}P_2 - \frac{7}{3}\right)2P_3 + 3\left(\frac{1}{2}P - \alpha + \frac{1}{2}P_2 + \frac{2}{9}\right)^2 + \frac{1}{2}\left(\frac{3}{2}P_2 - 3\alpha + \frac{2}{3}\right)^2 - \frac{9}{8}P^2$ and the first order condition with respect to P_3, at $P_3 = P$, is $\frac{57}{8}\left(P - (P_2 + \frac{4}{9} - 2\alpha)\right)\left(P - (\frac{5}{19}P_2 + \frac{20}{171} - \frac{10}{19}\alpha)\right) = 0$. Combining this with (28) (for the case of $P + 2\alpha > \frac{31}{90}$) yields P_2^*, P^* in Lemma 8(ii).

[51] Solving the system consisting of $\frac{9}{8}P^2 - 3P - \frac{9}{2}\alpha^2 + \frac{9}{2}\alpha P_2 - 2\alpha - \frac{9}{8}P_2^2 + P_2 + \frac{1}{3} = 0$ and (28) leads to the following third degree equation in P: $-3P^3 + (\frac{53}{6} - 3\alpha)P^2 + (6\alpha^2 + \frac{59}{18}\alpha - \frac{226}{81})P - \frac{10}{9}\alpha^2 - \frac{40}{81}\alpha + \frac{227}{972} = 0$, for which no solution can be expressed in terms of real radicals. For this reason $\rho(\alpha)$ is a complex number, although P^* and P_2^* in (60)-(61) are real numbers.

7.6 Profit comparisons for the study of the reduced game in Γ_α: (33)-(36)

Here we report the plots of the profit functions linked to (33)-(36):

Π_2^{123}, solid curve, vs Π_2^3, dashed curve: see (33) (Figure 36)

Π_3^3, solid curve, vs $\Pi_3^{\varnothing}$, dashed curve: see (34) (Figure 37)

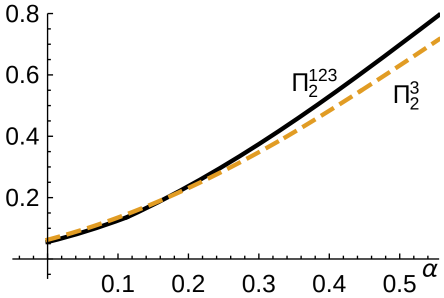


Figure 36

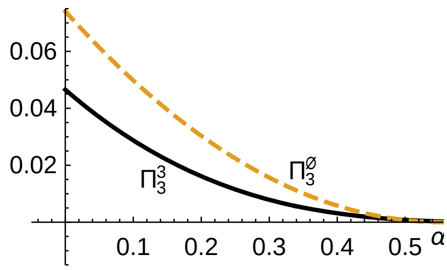


Figure 37

Π_3^{123}, solid curve, vs Π_3^1, dashed curve: see (35) (Figure 38)

Π_3^{123}, solid curve, vs Π_3^2, dashed curve: see (36) (Figure 39)

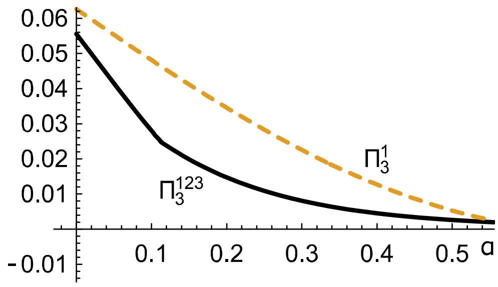


Figure 38

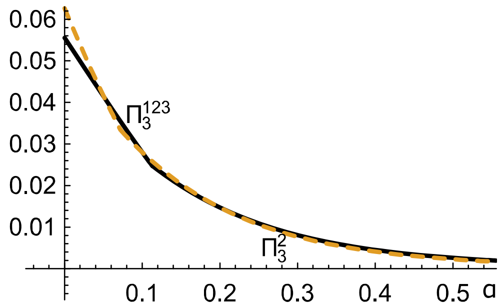


Figure 39

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