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A Dynamic Exchange Rate Model with Heterogeneous Agents*

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Abstract

In this paper, we analyze a heterogeneous agent model in which the fundamental exchange rate is endogenously determined by the real markets. The exchange rate market and the real markets are linked through the balance of payments. We have analytically found that there exists at least a steady state in which the exchange rate is at its fundamental value and incomes of both countries are equal to the autonomous components times the over-simplified multiplier (as in the Income-Expenditure model). That steady state can be unique and always unstable when all agents act as contrarians, while when agents act as fundamentalists is unique but its stability depends on the reactivity of actors of the market. Finally, we show that the (in)stability of the economic system depends on both the reactivity of the markets and that of different type of agents involved.

Keywords: Complex Dynamics; Heterogeneous Agents Models; Financial Markets.

JEL classification: C62, D84, E12, E32, G02

1 Introduction

In the last decades, models of financial markets have shown how complex dynamics of price fluctuations are related to the interactions between heterogeneous agents. Heterogeneity is either related to the strategies applied (fundamentalists, chartists, contrarians, noise traders) or may emerge in the beliefs about the fundamental value (see for surveys Hommes, 2006, LeBaron 2006 and Westerhoff, 2009).

In general, there is a lot of uncertainty on what the "true fundamental value" is. This is due to a strong subjective dimension of the estimation procedure: on the one hand agents do

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not necessarily follow the same 'structural model of the world', estimating the fundamental value in different ways; on the other hand, even when they use the same model, agents may hardly reach the same expectations because of different computational skills. The latter kind of heterogeneity is already incorporated in Brock and Hommes (1998) that contains a 3-type asset-pricing model with fundamentalists, upward and downward biased traders. We could interpret this example as a model with three groups of distinct fundamentalists, one with the correct estimate of the fundamental and two other types with a wrong estimate of the fundamental. In this line of research, De Grauwe and Rovira (2012) and Rovira (2010) study the effect of *biased traders* (optimists or pessimists) in the exchange market, while Naimzada and Ricchiuti (2008, 2009) have developed a two-type version of biased traders model, with a market maker structure, in which the source of instability resides in the interaction of fundamentalists with heterogeneous beliefs about the fundamental value of the asset: complex dynamics arise for a great distance in beliefs.

A key shortcoming of this literature is that the fundamental value is either fixed (as in the cases cited above) or time varying (as in Anufriev, 2013) but, as stated by De Grauwe and Rovira (2012), *'it still is exogenously determined. That is, it is not connected to the real part of the economy in any way'*. To the best of our knowledge only few papers have dealt with an endogenous fundamental value, in particular Lengnick and Wohltmann (2011), Proaño (2011), Westerhoff (2012) and Naimzada and Pireddu (2014a, 2014b).

Lengnick and Wohltmann (2011) combine an agent based model with a new-keynesian model. In their model, the fundamental value is fixed but it is assumed that the expectations vary over time following the unconditional standard deviation of the output gap. Proaño (2011) analyzes a two country macroeconomic model using a heterogeneous agents approach for the determination of the nominal exchange rate. His model has both chartists and fundamentalists and the fundamental value depends on the purchasing power parity. On the other hand, Westerhoff (2012) linked a stock market with heterogeneous speculators with a Keynesian goods market model for a closed economy. He assumes that the performance of the stock market affects both consumption and investment and, at the same time, is affected by the national income. Naimzada and Pireddu (2013a) introduce two changes to the Westerhoff's model: firstly, they use of a nonlinear adjustment for the real market and, secondly, they assume agents to make their decision using a linear combination between an exogenous and an endogenous value of both national income and level of stock market. The parameter used to determinate the weighted average is a proxy of market segmentation and, therefore, may be viewed as a policy that reduces the feed-back effects between markets.

In line with these models and recalling the De Grauwe and Rovira's (2012) suggestion, we propose and analyze a *purely qualitative* (Federici and Gandolfo, 2012) and discrete time model of the exchange rate model in which the long-run (fundamental) value of the exchange rate endogenously depends on the balance of payment. We combine a simple agent based model for the exchange rate (similar to those developed by De Grauwe and Rovira and Naimzada and Ricchiuti cited above) and the Keynesian goods market for two Countries (the Absorbtion Model)[1].

We have analytically found that there is at least a steady state in which the exchange rate is equal to its fundamental value, that is, the exchange rate for which the balance of payment is equal to zero, and incomes of both countries are equal to the autonomous components times the over-simplified multiplier: they do not depend on the marginal propensity to import, so that a flexible exchange rate, at this equilibrium, can isolate the two economic systems. Moreover, the steady state is unique and unstable when all agents act as contrarians, and it is unique and may be unstable when all agents act as fundamentalists depending on the

[1]Our model is close to that developed by Laursen and Metzler in 1950. The Laursen and Metzler model (1950) is static, while its continuous time dynamic version can be found in Gandolfo (1986).

reactivity of markets and agents.

2 Set up of the model

There are two countries (Europe and USA), who freely trade without restrictions and who have both a flexible exchange rate regime. As in Laursen and Metzler (1950) we assume that *'foreign exchange may be purchased and sold freely for all purposes except capital movement'*: that means that capital transactions are restricted and strictly related to the operations in exports and imports. Therefore the balance of payment depends only on the current account. We know that nowadays it is a strong assumption, but to leave the model as simple as possible, we prefer to relax it in a further work as we will discuss in the conclusions.

Therefore, the two Keynesian good markets are linked only through imports and exports, whose values depend on the exchange rate. We firstly present the exchange rate determination and then the two Keynesian good markets.

2.1 Exchange Rate Determination

Let $E(t)$ be the exchange rate at time t between the two countries, expressed as dollars per euros. The actual exchange rate is determined by a market maker who looks at the excess of demand of euros in the system, so that exchanges are possible out of equilibrium. The law of motion of the exchange rate is

$$E(t+1) = E(t) + g_3(ED(t)), \tag{1}$$

where g_3 is a C^1 function defined on $\mathbb{R}$ such that $g_3(0) = 0$ and $g_3'(s) > 0$ for all $s \in \mathbb{R}$ which describes the adjustment of the market maker with respect to the excess of demand $ED(t)$. In line with the Heterogenous Agents Models, we assume that there are agents with heterogenous strategies (fundamentalists vs. contrarians, the difference will be explained below) and/or beliefs about the fundamental value. Moreover, similarly to Kirman (1998), our model involves agents who act on the currency market on the basis of the forecast developed by *gurus/experts* who have developed their own mechanism to predict the future exchange rate, based on the current exchange rate and its fundamental value. Their suggestion is a *focal point* for the agents.

Particularly, given that the balance of payment is the difference between exports and imports, and that exports of country 1 (Europe) are the imports of country 2 (USA) and vice versa, we have that the balance of payment at time t is equal to zero when

$$\frac{mpi_2 Y_2(t)}{E(t)} - mpi_1 Y_1(t) = 0,$$

where $Y_1(t)$ is the income of country 1 at time t, $Y_2(t)$ the income of country 2 at time t (both expressed in their own currency), and $mpi_i \in (0,1)$ is the marginal propensity to import of country i for all $i \in \{1,2\}$. Then the fundamental value $\tilde{F}(t)$ of the exchange rate at time t is defined as[2]

$$\tilde{F}(t) = \frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}.$$

We assume there are n gurus, where $n \in \mathbb{N}$, and we suppose that, for every $j \in \{1,\ldots,n\}$, the guru j determines at time $t$her forecasting on the exchange rate at time $t+1$, say $\tilde{E}_j(t)$,

[2] As in De Arcangelis and Gandolfo (1997), we do not adhere to the traditional approach to the exchange rate, but the determination of the exchange rate is 'reflected in the balance of payment equation', which summarize the foreign exchange market.

looking at the ratio $\tilde{F}(t)$ and the exchange rate $E(t)$ and computing

$$\tilde{E}_j(t) = E(t) + k_j\left(\tilde{F}(t), E(t)\right) = E(t) + k_j\left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t)\right)$$

where $k_j : \mathbb{R}^2_{++} \to \mathbb{R}$ is a C^1 function such that, for every $s \in \mathbb{R}$, $k_j(s, s) = 0$. The function k_j incorporates then all the skills that guru j has to figure out how the exchange rate will vary from a time period to the next one on the basis of the current exchange rate and its current fundamental value. Note that if guru j observes that the current exchange rate equals its fundamental value, that is, $\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)} = E(t)$, then she believes that in the future the exchange rate will not change. According to the related literature on the topic, we also say that guru j is fundamentalist if

$$\text{for every } (z_1, z_2) \in \mathbb{R}^2_{++},\ \text{sign}(k_j(z_1, z_2)) = \text{sign}(z_1 - z_2), \tag{2}$$

while we say she is contrarian if

$$\text{for every } (z_1, z_2) \in \mathbb{R}^2_{++},\ \text{sign}(k_j(z_1, z_2)) = -\text{sign}(z_1 - z_2). \tag{3}$$

We also denote by $k : \mathbb{R}^2_{++} \to \mathbb{R}^n$ the function such that, for every $(z_1, z_2) \in \mathbb{R}^2$, $k(z_1, z_2) = (k_j(z_1, z_2))_{j=1}^n$.

Moreover, financial operators carry out their demand of currency based on the suggestions of a specific guru j, specifically, the agents that follow the guru j at time t are $\pi_j(t)$. In this literature quotas may be either fixed (Day and Huang, 1990) or variable; in the latter case agents can switch from one expert to an other one, following for example an adaptive belief system (as in Brock and Hommes, 1998). The switching mechanisms are usually based either on profitability (the agents choose the trading rule or the guru according to its capacity to make profits) or on fitness (agents choose the rule or the guru that indicate the closer proximity to the actual price). We do not assume a specific functional form for the switching mechanism, requiring just that, for every $j \in \{1, \dots, n\}$,

$$\pi_j(t) = \lambda_j\left(k\left(\tilde{F}(t), E(t)\right)\right),$$

where $\lambda_j : \mathbb{R}^n \to (0, 1)$ is C^1 function and, for every $x \in \mathbb{R}^n$, $\sum_{j=1}^n \lambda_j(x) = 1$. Therefore, the quotas may vary on time depending on the forecastings of all the gurus. For instance, agents who do not believe in the possibility of strong leaps of the variation of the exchange rate may follow that guru who forecasts the smallest change in the exchange rate.

Finally, the market maker operates adjusting the price according to the aggregate excess demand of euros, that is,

$$E(t+1) = E(t) + g_3\left(\sum_{j=1}^n \lambda_j\left(k\left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t)\right)\right)\gamma k_j\left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t)\right)\right), \tag{4}$$

where $\gamma > 0$ and g_3 is the function in (1). The constant γ transforms dollars in euros and represents the reactivity of agents, so that, or every $j \in \{1, \dots, n\}$, the quantity

$$\gamma k_j\left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t)\right)$$

represents the excess of demand of any single agent who adheres to the guru j. Then

$$\sum_{j=1}^n \lambda_j\left(k\left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t)\right)\right)\gamma k_j\left(\frac{mpi_2 Y_2(t)}{mpi_1 Y_1(t)}, E(t)\right)$$

represents the aggregate excess of demand of euros per person. Given the effect on the balance of trade, the exchange rate affects the GDP of both countries which afterwards have a feedback effect on the exchange rate through its fundamental value.

2.2 Absorbtion Model

From the income-expenditure model, we know that a generic demand of goods is given by the sum of expenditures (consumptions, investments, government expenditure and net export), that is,

$$D = \bar{C} + I + G + X - \bar{Z} = C + mpcY + I + G + X - mpiY = A + mpcY + X - mpiY \quad (5)$$

where A is the sum of the autonomous components of demand the $C + I + G$, mpc is the marginal propensity to consume, and mpi is the marginal propensity to import. In what follows, A is assumed to be positive and $0 < mpi < mpc < 1$. As stated above, the exports of one country are the imports of the other country, so that in each period the demand/expenditure of European citizens in euros is

$$D_1(t) = A_1 + mpc_1Y_1(t) + \frac{mpi_2Y_2(t)}{E(t)} - mpi_1Y_1(t) \quad (6)$$

and that of US citizens in dollars is

$$D_2(t) = A_2 + mpc_2Y_2(t) + mpi_1Y_1(t)E(t) - mpi_2Y_2(t). \quad (7)$$

Moreover, for every $i \in \{1, 2\}$, we assume that the GDP of country i at time $t + 1$ depends on its previous value and the excess of demand at time t as follows[3]

$$Y_i(t + 1) = Y_i(t) + g_i(D_i(t) - Y_i(t)) \quad (8)$$

where g_i is a C^1 function defined on $\mathbb{R}$ such that $g_i(0) = 0$ and $g_i'(s) > 0$ for all $s \in \mathbb{R}$ [4]. Please note that, with a fixed exchange rate we would achieve the well-known absorbtion model studied in a normal course of International Economics. Moreover, the model is close to that developed by Laursen and Metzler (1950).

By (4) and (8), we are then left with analysing the discrete dynamical system:

$$\begin{cases} Y_1(t+1) = Y_1(t) + g_1\left(A_1 + mpc_1Y_1(t) + \frac{mpi_2Y_2(t)}{E(t)} - mpi_1Y_1(t) - Y_1(t)\right) \\ Y_2(t+1) = Y_2(t) + g_2\Big(A_2 + mpc_2Y_2(t) + mpi_1Y_1(t)E(t) - mpi_2Y_2(t) - Y_2(t)\Big) \\ E(t+1) = E(t) + g_3\left(\sum_{j=1}^{n} \lambda_j \left(k\left(\frac{mpi_2Y_2(t)}{mpi_1Y_1(t)}, E(t)\right)\right) \gamma k_j \left(\frac{mpi_2Y_2(t)}{mpi_1Y_1(t)}, E(t)\right)\right) \end{cases} \quad (9)$$

where the functions Y_1, Y_2 and E are assumed to be positive.

3 Dynamic analysis

Consider the functions $r_1 : \mathbb{R}_{++} \to \mathbb{R}_{++}$ defined, for every $s \in \mathbb{R}_{++}$, as

$$r_1(s) = \frac{A_1(1 + mpi_2 - mpc_2)mpi_1s + A_2mpi_1mpi_2}{(1 + mpi_1 - mpc_1)(1 + mpi_2 - mpc_2)s - mpi_1mpi_2s},$$

[3]Obviously the demand can absorb all the product. However, when there is an excess of demand, we assume that both countries have to ration all components of the demand other than export.

[4]In Westerhoff (2012) that function is linear while for Naimzada and Pireddu (2013a) it is sigmoidal.

and $r_2 : \mathbb{R}_{++} \to \mathbb{R}_{++}$ defined, for every $s \in \mathbb{R}_{++}$, as

$$r_2(s) = \frac{A_2(1 + mpi_1 - mpc_1)mpi_2 + A_1 mpi_1 mpi_2 s}{(1 + mpi_1 - mpc_1)(1 + mpi_2 - mpc_2) - mpi_1 mpi_2}.$$

The following proposition is our main result. Its proof is in the appendix.

Proposition 1. *The set of steady states of the system* (9) *is given by*

$$S = \left\{ \left(\frac{r_1(E)}{mpi_1}, \frac{r_2(E)}{mpi_2}, E \right) \in \mathbb{R}^3_{++} : E \in \Gamma \right\}$$

where

$$\Gamma = \left\{ E \in \mathbb{R}_{++} : \sum_{j=1}^{n} \lambda_j \left(k \left(\frac{r_2(E)}{r_1(E)}, E \right) \right) k_j \left(\frac{r_2(E)}{r_1(E)}, E \right) = 0 \right\}. \quad (10)$$

In particular,

$$(Y_1^*, Y_2^*, E^*) = \left(\frac{A_1}{1 - mpc_1}, \frac{A_2}{1 - mpc_2}, \frac{A_2\, mpi_2\, (1 - mpc_1)}{A_1\, mpi_1\, (1 - mpc_2)} \right) \quad (11)$$

belongs to S *and is the unique steady state if, for every* $(z_1, z_2) \in \mathbb{R}^2_{++}$,

$$\text{sign}(k_1(z_1, z_2)) = \ldots = \text{sign}(k_n(z_1, z_2)).$$

Moreover, defining

$$R = g_3'(0)\gamma \sum_{j=1}^{n} \lambda_j(0) \frac{\partial k_j}{\partial z_1} (E^*, E^*), \quad (12)$$

we have that (11) *is unstable if one of the following set of conditions holds true:*

A1) $R < 0$;

A2) $\frac{2}{1-mpc_1+mpi_1} < g_1'(0) < \frac{2}{1-mpc_1}$, $g_2'(0) < \frac{2}{1-mpc_2+mpi_2}$, $R > 0$;

A3) $g_1'(0) < \frac{2}{1-mpc_1+mpi_1}$, $\frac{2}{1-mpc_2+mpi_2} < g_2'(0) < \frac{2}{1-mpc_2}$, $R > 0$;

A4) $g_1'(0) > \frac{4}{1-mpc_1}$, $g_2'(0) > \frac{4}{1-mpc_2}$, $R > 2\left(1 + \frac{2mpi_1}{1-mpc_1}\right)\left(1 + \frac{2mpi_2}{1-mpc_2}\right)$.

Finally, assuming $mpc_1 = mpc_2$, $mpi_1 = mpi_2$, $g_1'(0) = g_2'(0)$, *and* $R > 0$ *we have that*

B1) if $g_1'(0) \geq \frac{2}{1-mpc_1}$, *then* (11) *is unstable;*

B2) if $g_1'(0) < \frac{2}{1-mpc_1}$ *and* $\frac{R}{2} > \frac{1+mpc_1-2mpi_1}{1+mpc_1}$, *then* (11) *is unstable;*

B3) if $g_1'(0) < \frac{2}{1-mpc_1}$ *and* $\frac{R}{2} < \frac{1+mpc_1-2mpi_1}{1+mpc_1}$, *then* (11) *is asymptotically stable.*

Given the mild assumptions on the functions involved in the model, it is straightforward that multiple equilibria may arise for many specifications of them. Among the steady states there is always (11) which has some interesting economic properties. First of all, differently from Laursen and Metzler (1950), the steady state value of the Income of each country does not depend neither on the variation of the Exchange Rate nor on the marginal propensity to import, but only from their marginal propensity to consume. This means that, at that equilibrium, the flexible exchange rate can isolate the two economic systems and that an increase of Income in country 1 does not lead to an increase of Income in country 2, as suggested by the Absorption Model: the variation are absorbed by the fluctuations of the exchange rate.

Moreover, the exchange rate in (11) is the value that guarantees the equilibrium of the Balance of Payment[5]. Differently from Naimzada and Ricchiuti (2009) and De Grauwe and Rovira (2012), at that *fundamental* steady state if the gurus observe that the exchange rate is equal the fundamental, they believe that in the future the exchange rate will not change. In a world where the determination of the exchange rate completely depends on speculative agents, it is possible that a belief around a fundamental leads to its self-fulfill. It is worth noting that in particular (11) is unique if all gurus act as fundamentalists (contrarians).

From a stability point of view, we find several sufficient conditions for either stability or instability. These conditions involve the parameters R, $g_1'(0)$ and $g_2'(0)$ and marginal propensities. Note that R takes into account the reactivity of agents (via γ), gurus (via k) and exchange market (via $g_3'(0)$), while the parameters $g_1'(0)$ and $g_2'(0)$ represents the reactivity of good markets. In particular, under the assumption that there exists $j^* \in \{1, \dots, n\}$ such that

$$\frac{\partial k_{j^*}}{\partial z_1}(E^*, E^*) \neq 0, \tag{13}$$

we get instability when all the gurus are contrarians (see A1). Similarly to Naimzada and Ricchiuti (2008, 2009), instability can also be obtained when all the gurus are fundamentalists, provided the reactivity of gurus, agents and markets are high enough (see A4). Finally, under the assumption that the two countries are equal and all gurus are fundamentalists, we get stability when the good market reactivity is less than twice the well-known oversimplified multiplier and all the other reactivities are low enough.

4 Conclusions

We develop a heterogenous agents model in which the fundamental value of the exchange rate is endogenous, depending on the balance of payments and through it gives and receives feed-backs from the real markets. In this way, we overcome the exogenous determination of fundamentals, one of the most important shortcomings in the literature on heterogeneous agents models. Other authors have recently worked on this topic. Specifically, our work is strictly related to that of Westerhoff (2012) and Naimzada and Pireddu (2013a, 2013b). However, differently from these papers, we propose a model for the exchange rate determination, linking the exchange market with goods market of two countries as in the Absorption model.

We believe that further researches have to overcome some issues of our paper. First of all, even though our main aim here was to build a framework with an endogenous fundamental reducing complexity at minimum, in the foreign market we neglect the interesting case of gurus who are trend followers or noise traders. Secondly, as mentioned above, free capital transactions should be introduced to analyse the effects of the financial sector on both foreign and good markets. However, this entails a broader and more complex model with the addition of (at least) the money market. Finally, we believe that a problem in this kind of models is the definition of time. We assumed along the paper that variations of both the exchange rate and incomes take place at the same moment: this means that our is necessarily a medium-run model. However, in the heterogeneous agents model literature the variations of the exchange rate are usually assumed intra-daily. To overcome this issue, we believe that it should be studied a hybrid model in which the variations happen in different time scale.

[5]This is the only steady state having that property.

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A Proof of Proposition 1

Defining $y_1(t) = mpi_1 Y_1(t)$ and $y_2(t) = mpi_2 Y_2(t)$, we can write system (9) as follows

$$\begin{cases} y_1(t+1) = y_1(t) + mpi_1 g_1 \left(A_1 + \frac{mpc_1}{mpi_1} y_1(t) + \frac{y_2(t)}{E(t)} - y_1(t) - \frac{1}{mpi_1} y_1(t) \right) \\ y_2(t+1) = y_2(t) + mpi_2 g_2 \left(A_2 + \frac{mpc_2}{mpi_2} y_2(t) + y_1(t)E(t) - y_2(t) - \frac{1}{mpi_2} y_2(t) \right) \\ E(t+1) = E(t) + g_3 \left(\gamma \sum_{j=1}^n \lambda_j \left(k\left(\frac{y_2(t)}{y_1(t)}, E(t) \right) \right) k_j \left(\frac{y_2(t)}{y_1(t)}, E(t) \right) \right) \end{cases}$$

Defining now

$$\begin{aligned} & a_1 = A_1 > 0, \quad a_2 = A_2 > 0 \\ & b_1 = -\frac{mpc_1}{mpi_1} + 1 + \frac{1}{mpi_1} > 1, \quad b_2 = -\frac{mpc_2}{mpi_2} + 1 + \frac{1}{mpi_2} > 1 \\ & f_1 = mpi_1 g_1, \quad f_2 = mpi_2 g_2, \quad f_3 = g_3 \end{aligned} \tag{14}$$

we get the following system

$$\begin{cases} y_1(t+1) = y_1(t) + f_1 \left(a_1 - b_1 y_1(t) + \frac{y_2(t)}{E(t)} \right) \\ y_2(t+1) = y_2(t) + f_2 \Big(a_2 - b_2 y_2(t) + y_1(t)E(t) \Big) \\ E(t+1) = E(t) + f_3 \left(\gamma \sum_{j=1}^n \lambda_j \left(k\left(\frac{y_2(t)}{y_1(t)}, E(t) \right) \right) k_j \left(\frac{y_2(t)}{y_1(t)}, E(t) \right) \right) \end{cases} \tag{15}$$

Denoting by $\widehat{S}$ the set of steady states of (15), we have that the function $\Phi : \widehat{S} \to S$ mapping $(y_1, y_2, E) \in \widehat{S}$ into $(y_1/mpi_1, y_2/mpi_2, E) \in S$ is well defined and is a bijection. Moreover, $(y_1, y_2, E) \in \widehat{S}$ is asymptotically stable if and only if $\Phi(y_1, y_2, E) \in S$ is asymptotically stable, and $(y_1, y_2, E) \in \widehat{S}$ is unstable if and only if $\Phi(y_1, y_2, E) \in S$ is unstable. As a consequence, we can prove Proposition 1 about system (9) working with system (15) and properly using the bijection Φ and equalities (14).

Note first that, for every $s \in \mathbb{R}_{++}$,

$$r_1(s) = \frac{a_1 b_2 s + a_2}{(b_1 b_2 - 1)s} \quad \text{and} \quad r_2(s) = \frac{a_2 b_1 + a_1 s}{b_1 b_2 - 1}.$$

Since the system

$$\begin{cases} a_1 - b_1 y_1 + \frac{y_2}{E} = 0 \\ a_2 - b_2 y_2 + y_1 E = 0 \\ y_1, y_2, E > 0 \end{cases} \tag{16}$$

has a set of solutions given by

$$\{(r_1(E), r_2(E), E) \in \mathbb{R}^3_{++} : E > 0\},$$

we have that

$$\widehat{S} = \{(r_1(E), r_2(E), E) \in \mathbb{R}^3_{++} : E \in \Gamma\},$$

where Γ is defined in (10), and that immediately implies the first statement of Proposition 1. Note also that

$$E^* = \frac{a_2(b_1-1)}{a_1(b_2-1)} \in \Gamma.$$

Then

$$(y_1^*, y_2^*, E^*) = \left(\frac{a_1}{b_1-1}, \frac{a_2}{b_2-1}, \frac{a_2(b_1-1)}{a_1(b_2-1)}\right), \tag{17}$$

belongs to $\widehat{S}$ and since $\Phi(y_1^*, y_2^*, E^*)$ is equal to (11) and belongs to S, we get the second statement of Proposition 1.

Assume now that, for every $(z_1, z_2) \in \mathbb{R}^2_{++}$, we have that

$$\text{sign}(k_1((z_1, z_2))) = \ldots = \text{sign}(k_n((z_1, z_2))),$$

and prove that $\widehat{S} = \{(y_1^*, y_2^*, E^*)\}$, so that $S = \{\Phi(y_1^*, y_2^*, E^*)\}$. Indeed,

$$\sum_{j=1}^{n} \lambda_j \left(k\left(\frac{r_2(E)}{r_1(E)}, E\right)\right) k_j \left(\frac{r_2(E)}{r_1(E)}, E\right) = 0$$

if and only if

$$\frac{r_2(E)}{r_1(E)} - E = 0,$$

that is,

$$\frac{a_2(b_1-1)E - a_1(b_2-1)E^2}{a_2 + a_1 b_2 E} = 0$$

and that equation has E^* has unique solution. That proves the third part of Proposition 1.

Let us move on to prove the last part of the proposition. We analyse then the stability of the state (y_1^*, y_2^*, E^*) defined in (17) in order to deduce information about the stability of $\Phi(y_1^*, y_2^*, E^*)$, that is, (11). Consider the three following functions whose domain is $\mathbb{R}^3_{++}$,

$$F_1(y_1, y_2, E) = y_1 + f_1\left(a_1 - b_1 y_1 + \frac{y_2}{E}\right)$$

$$F_2(y_1, y_2, E) = y_2 + f_2\Big(a_2 - b_2 y_2 + y_1 E\Big)$$

$$F_3(y_1, y_2, E) = E + f_3\left(\gamma \sum_{j=1}^{n} \lambda_j \left(k\left(\frac{y_2}{y_1}, E\right)\right) k_j \left(\frac{y_2}{y_1}, E\right)\right)$$

Then we have that:

$$\begin{aligned}
\tfrac{\partial F_1}{\partial y_1}\left(y_1^*, y_2^*, E^*\right) = & \quad 1 - b_1 f_1'(0) \\
\tfrac{\partial F_1}{\partial y_2}\left(y_1^*, y_2^*, E^*\right) = & \quad \tfrac{a_1(b_2-1)}{a_2(b_1-1)} f_1'(0) \\
\tfrac{\partial F_1}{\partial E}\left(y_1^*, y_2^*, E^*\right) = & \quad -\tfrac{a_1^2(b_2-1)}{a_2(b_1-1)^2} f_1'(0) \\
\tfrac{\partial F_2}{\partial y_1}\left(y_1^*, y_2^*, E^*\right) = & \quad \tfrac{a_2(b_1-1)}{a_1(b_2-1)} f_2'(0) \\
\tfrac{\partial F_2}{\partial y_2}\left(y_1^*, y_2^*, E^*\right) = & \quad 1 - b_2 f_2'(0) \\
\tfrac{\partial F_2}{\partial E}\left(y_1^*, y_2^*, E^*\right) = & \quad \tfrac{a_1}{(b_1-1)} f_2'(0) \\
\tfrac{\partial}{\partial y_1} F_3\left(y_1^*, y_2^*, E^*\right) = & \quad -f_3'(0)\tfrac{a_2(b_1-1)^2}{a_1^2(b_2-1)}\gamma \textstyle\sum_{j=1}^n \lambda_j(0)\tfrac{\partial k_j}{\partial z_1}\left(\tfrac{a_2(b_1-1)}{a_1(b_2-1)}, \tfrac{a_2(b_1-1)}{a_1(b_2-1)}\right) \\
\tfrac{\partial F_3}{\partial y_2}\left(y_1^*, y_2^*, E^*\right) = & \quad f_3'(0)\tfrac{(b_1-1)}{a_1}\gamma \textstyle\sum_{j=1}^n \lambda_j(0)\tfrac{\partial k_j}{\partial z_1}\left(\tfrac{a_2(b_1-1)}{a_1(b_2-1)}, \tfrac{a_2(b_1-1)}{a_1(b_2-1)}\right) \\
\tfrac{\partial F_3}{\partial E}\left(y_1^*, y_2^*, E^*\right) = & \quad 1 + f_3'(0)\gamma \textstyle\sum_{j=1}^n \lambda_j(0)\tfrac{\partial k_j}{\partial z_2}\left(\tfrac{a_2(b_1-1)}{a_1(b_2-1)}, \tfrac{a_2(b_1-1)}{a_1(b_2-1)}\right)
\end{aligned}$$

Note that

$$R = f_3'(0)\gamma \sum_{j=1}^n \lambda_j(0)\frac{\partial k_j}{\partial z_1}\left(\frac{a_2(b_1-1)}{a_1(b_2-1)}, \frac{a_2(b_1-1)}{a_1(b_2-1)}\right),$$

where R is defined in (12), and, for every $j \in \{1, \ldots, n\}$ and $s \in \mathbb{R}$,

$$\frac{\partial k_j}{\partial z_1}(s, s) = -\frac{\partial k_j}{\partial z_2}(s, s).$$

Then we have that

$$\begin{aligned}
\frac{\partial F_3}{\partial y_1}\left(y_1^*, y_2^*, E^*\right) &= -\frac{a_2(b_1-1)^2}{a_1^2(b_2-1)} R \\
\frac{\partial F_3}{\partial y_2}\left(y_1^*, y_2^*, E^*\right) &= \frac{(b_1-1)}{a_1} R \\
\frac{\partial F_3}{\partial E}\left(y_1^*, y_2^*, E^*\right) &= R
\end{aligned}$$

In order to study the stability of (17), we study the stability of the polinomial

$$P(s) = \det\left(sI - D(F_1, F_2, F_3)\left(y_1^*, y_2^*, E^*\right)\right) =$$

$$\det \begin{bmatrix} s - 1 + b_1 f_1'(0) & -\frac{a_1(b_2-1)}{a_2(b_1-1)} f_1'(0) & \frac{a_1^2(b_2-1)}{a_2(b_1-1)^2} f_1'(0) \\ -\frac{a_2(b_1-1)}{a_1(b_2-1)} f_2'(0) & s - 1 + b_2 f_2'(0) & -\frac{a_1}{(b_1-1)} f_2'(0) \\ \frac{a_2(b_1-1)^2}{a_1^2(b_2-1)} R & -\frac{(b_1-1)}{a_1} R & s - 1 + R \end{bmatrix}$$

that is, we study whether all of its roots lie inside the unit circle or there is at least one root lying outside the unit circle. A computation shows that

$$P(s) = (s-1)^3 + \rho_1(s-1)^2 + \rho_2(s-1) + \rho_3$$

where

$$\rho_1 = R + b_2 f_2'(0) + b_1 f_1'(0),$$

$$\rho_2 = R b_2 f_2'(0) + R b_1 f_1'(0) + b_1 b_2 f_1'(0) f_2'(0) - f_1'(0) R - f_1'(0) f_2'(0) - f_2'(0) R,$$

$$\rho_3 = R f_1'(0) f_2'(0)(b_1 b_2 - b_1 - b_2 + 1).$$

A simple condition which is sufficient for instability is $P(1) < 0$, that is, $\rho_3 < 0$. Since each factor in $f_1'(0)f_2'(0)(b_1b_2 - b_1 - b_2 + 1)$ is positive, we have that $P(1) < 0$ if and only if $R < 0$. That proves Statement A1 of Proposition 1.

An other simple condition for instability is $P(-1) > 0$, that is, $-8 + 4\rho_1 - 2\rho_2 + \rho_3 > 0$. Note that,

$$-8 + 4\rho_1 - 2\rho_2 + \rho_3 > 0$$

if and only if

$$R\Big[4 - 2b_2 f_2'(0) - 2b_1 f_1'(0) + 2f_2'(0) + 2f_2'(0) + f_1'(0)f_2'(0)(b_1b_2 - b_1 - b_2 + 1)\Big] > \tag{18}$$

$$8 - 4b_2 f_2'(0) - 4b_1 f_1'(0) + 2f_1'(0)f_2'(0)b_1b_2 - 2f_1'(0)f_2'(0)$$

Since

$$4 - 2b_2 f_2'(0) - 2b_1 f_1'(0) + 2f_2'(0) + 2f_2'(0) + f_1'(0)f_2'(0)(b_1b_2 - b_1 - b_2 + 1) =$$

$$\Big(2 - f_1'(0)(b_1 - 1)\Big)\Big(2 - f_2'(0)(b_2 - 1)\Big),$$

and $f_1'(0), f_2'(0) > 0$, we have that (18) is implied by

$$R\Big(2 - f_1'(0)(b_1 - 1)\Big)\Big(2 - f_2'(0)(b_2 - 1)\Big) > 8 - 4b_2 f_2'(0) - 4b_1 f_1'(0) + 2f_1'(0)f_2'(0)b_1b_2$$

that is

$$R\Big(2 - f_1'(0)(b_1 - 1)\Big)\Big(2 - f_2'(0)(b_2 - 1)\Big) > 2\Big(2 - f_1'(0)b_1\Big)\Big(2 - f_2'(0)b_2\Big). \tag{19}$$

It can be verified that (19) holds true if one of the following set of conditions is satisfied:

- $\frac{2}{b_1} < f_1'(0) < \frac{2}{b_1 - 1}$, $f_2'(0) < \frac{2}{b_2}$, and $R > 0$;
- $f_1'(0) < \frac{2}{b_1}$, $\frac{2}{b_2} < f_2'(0) < \frac{2}{b_2 - 1}$, and $R > 0$;
- $f_1'(0) > \frac{4}{b_1 - 1}$, $f_2'(0) > \frac{4}{b_2 - 1}$, $R > 2\frac{(b_1+1)(b_2+1)}{(b_1-1)(b_2-1)}$.

Those conditions are exactly the ones in the Statements $A2$, $A3$ and $A4$ of Proposition 1.

In order to prove the last part of the proposition assume $mpc_1 = mpc_2$, $mpi_1 = mpi_2$, $g_1'(0) = g_2'(0)$, and $R > 0$. That implies $b_1 = b_2$ and $f_1'(0) = f_2'(0)$ and

$$P(s) = (s-1)\left[(s-1)^2 + 2bf(s-1) + f^2(b^2 - 1)\right] + R\left[(s-1)^2 + 2f(b-1)(s-1) + f^2(b-1)^2\right]$$

where we set $b = b_1$ and $f = f_1'(0)$. Let $P(s) = (s-1)K(s) + RJ(s)$, where

$$K(s) = (s-1)^2 + 2bf(s-1) + f^2(b^2 - 1),$$

and

$$J(s) = (s-1)^2 + 2f(b-1)(s-1) + f^2(b-1)^2.$$

The equation $K(s) = 0$ has three distinct solutions given by

$$s_1 = 1 - f(b+1), \quad s_2 = 1 - f(b-1), \quad 1,$$

with $s_1 < s_2 < 1$, and since $J(s) = (s-1) + f(b-1)]^2 = (s - s_2)^2$, we also have that $P(s_2) = 0$, $P'(s_2) < 0$, $P(s_1) > 0$ and, for every $s \in [1, +\infty)$, $P(s) > 0$. As a consequence P has three distinct real roots $s_1^* < s_2^* < s_3^*$ such that

$$s_1^* < s_1, \quad s_2^* = s_2, \quad s_3^* \in (s_2, 1).$$

As a consequence, if $f \geq \frac{2}{b+1}$, then $s_1^* < s_1 \leq -1$ and (17) is unstable: that proves Statement B1 of Proposition 1.

Assuming instead $f < \frac{2}{b+1}$, we get $s_1 > -1$ and then, $s_1^* < -1$ if and only if $P(-1) > 0$, while $s_1^* \in (-1, 1)$ if and only if $P(-1) < 0$.

A computation shows that

$$P(-1) = -8 + 8fb - 2f^2b^2 + 2f^2 + R(2 - f(b-1))^2,$$

and then $P(-1) > 0$ if and only if

$$R(2 - f(b-1))^2 > 8 - 8fb + 2f^2b^2 - 2f^2 = 2(2 - f(b+1))(2 - f(b-1))$$

if and only if

$$\frac{R}{2} > \frac{2 - f(b+1)}{2 - f(b-1)}.$$

Analogously we have that $P(-1) < 0$ if and only if

$$\frac{R}{2} < \frac{2 - f(b+1)}{2 - f(b-1)}.$$

In the first case we get instability while in the second case we get asymptotic stability: by a substitution we obtain conditions of Statements B2 and B3 of Proposition 1.