Dipartimento di Scienze Economiche
Università degli Studi di Firenze

Working Paper Series

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Working Paper N. 08/2011
November 2011

Dipartimento di Scienze Economiche, Università degli Studi di Firenze
Via delle Pandette 9, 50127 Firenze, Italia
www.dse.unifi.it

Stampato in proprio in Firenze dal Dipartimento Scienze Economiche
(Via delle Pandette 9, 50127 Firenze) nel mese di Maggio 2011,
Esemplare Fuori Commercio Per il Deposito Legale
agli effetti della Legge 15 Aprile 2004, N.106

Revenue comparison in asymmetric auctions with discrete valuations

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May 7, 2012

Abstract

We consider an asymmetric auction setting with two bidders such that the valuation of each bidder has a binary support. We prove that in this context the second price auction yields a higher expected revenue than the first price auction for a broad set of parameter values, although the opposite result is common in the literature on asymmetric auctions. For instance, when the probabilities of high values are the same, the second price auction is superior unless the distribution of a bidder's valuation first order stochastically dominates the distribution of the other bidder's valuation "in a strong sense". We prove that this result extends to some degree to the case of unequal probabilities, and to the case in which the valuation of each bidder is a three-point set. In addition, we show that in some cases the revenue in the first price auction decreases when all the valuations increase.

JEL Classification: D44, D82.

Key words: Asymmetric auctions, First price auctions, Second price auctions.

Acknowledgments: We thank René Kirkegaard and Andrey Sarychev for useful comments and suggestions. The usual disclaimer applies. This paper is part of a research project on Mechanism Design and Auctions which is financially supported by the Italian Ministry of the University.

1 Introduction

This paper is about a seller's preferences between a first price auction (FPA from now on) and a second price/Vickrey auction (SPA from now on) when the bidders' valuations are independently but asymmetrically distributed. Precisely, we consider a setting with two bidders such that the valuation of each bidder has a binary support (in our final section we consider supports including three points). In this environment we first derive the unique equilibrium outcome and the expected revenue in the FPA for all parameter values. Then we compare the revenue in the FPA with the revenue in the SPA. We prove that the SPA yields a higher revenue than the FPA for a broad set of parameter values, although the opposite result is common in the literature on asymmetric auctions (we provide an overview of this literature later on in this introduction). For instance, on the basis of numeric analysis for some classes of continuous distributions, Li and Riley (2007) claim that "the 'typical' case leads to greater expected revenue in the sealed high-bid auction" [i.e., in the FPA]; a similar point of view is found in Klemperer (1999).

More in detail, we use λ_1 (λ_2) to denote the probability of a low value for bidder 1 (bidder 2), and for the particular case in which $\lambda_1 = \lambda_2$ we find the following results.

- The revenue in the FPA may decrease when all the valuations increase, because increasing the high value of one bidder may induce his opponent to bid less aggressively. This makes the FPA inferior to the SPA.[1]
- The SPA is more profitable than the FPA for the seller if a bidder's valuation is more variable than the other bidder's valuation,[2] or if the distribution of a bidder's valuation first order stochastically dominates the distribution of his opponent's valuation – but not too strongly. Conversely, the FPA is superior to the SPA if the low value of a bidder is sufficiently larger than the high value of the other bidder.[3]

When $\lambda_1 \neq \lambda_2$ we show that several of the above results still hold, whereas others do not. Furthermore we show that the SPA dominates the FPA if the bidders' high values are the same.

Finally, we examine a particular setting in which each bidder's valuation has a three-point support, and for some small asymmetries we prove the same results we have obtained for binary supports when $\lambda_1 = \lambda_2$.

[1] This result contrasts with a claim in Maskin and Riley (1985) for the case in which the only deviation from a symmetric setting is given by unequal high valuations [this claim is reproduced in Klemperer (1999)]. However, for this case Maskin and Riley (1983) agree with our ranking between the FPA and the SPA

[2] After Vickrey (1961), this is the first ranking result in the theoretical literature which does not rely on first order stochastic dominance among the distributions of valuations.

[3] Doni and Menicucci (2011) study a procurement setting in which the auctioneer privately observes the qualities of the products offered by the suppliers and needs to decide how much of the own information on qualities should be revealed to suppliers before a (first score) auction is held. Our results on the comparison between the FPA and the SPA when $\lambda_1 = \lambda_2$ contribute to determining the best information revelation policy for the auctioneer.

In the rest of this introduction we provide an overview of the related literature. In Section 2 we describe the primitives of our model. In Section 3 we study equilibrium behavior in the SPA and in the FPA. In Section 4 we present our results on the comparison between the FPA and the SPA. Finally, in Section 5 we consider three-points supports. Sections 6-12 provide the proofs of our results.

Related literature The analysis of the FPA when the bidders' valuations are asymmetrically and continuously distributed is often difficult because the equilibrium bidding strategies are characterized by a system of differential equations (obtained from the first order condition for each type of bidder) which has a closed form solution only in very particular cases. For instance, Kaplan and Zamir (2010) derive (the inverse) equilibrium bidding functions under asymmetrically and uniformly distributed valuations; Plum (1992) and Cheng (2006) obtain closed form solutions for some special cases of power distributions.[4] Not surprisingly, matters are simpler if there are only two types for each bidder, rather than a continuum. Indeed, in such a case Maskin and Riley (1983) derive in closed form an equilibrium in mixed strategies under the assumption that the bidders' low values are coincident.[5] Proposition 1 in our paper extends this result, as we remove their assumption that the bidders' low valuations coincide.

As it is well known, with asymmetric distributions the revenue equivalence theorem does not apply, and the lack of a closed form for the equilibrium bidding functions complicates the comparison between the FPA and the SPA.[6] The known results show that there is not a general dominance of an auction format over the other, but the SPA has been proved to dominate the FPA mainly in some specific settings, whereas there exist results which establish the superiority of the FPA for a relatively broad set of circumstances, and not only for some particular examples. Precisely, Maskin and Riley (2000a) analyze a setting with continuously distributed valuations and show that the FPA is superior to the SPA if a bidder's valuation distribution satisfies suitable conditions (which include log-concavity) and the other bidder's valuation distribution is obtained by shifting or stretching to the right the first bidder's distribution. These results are obtained by examining the properties of the system of differential equations which characterize the equilibrium bidding strategies.

Kirkegaard (2011b) provides sufficient conditions for the FPA to dominate the SPA and his main theorem generalizes the results in Maskin and Riley (2000a) [see also Kirkegaard (2011a)]. He makes two main assumptions. The first one is that the distribution of the valuation of one bidder, the strong bidder, dominates the distribution for the other bidder, the weak bidder, in

[4] Cheng (2010) characterizes the auction environments such that each bidder's equilibrium bidding function is linear. He shows that this property requires that either each bidder's value distribution is a power function, or is the product of a power function and an exponential function.

[5] Cheng (2011) employs the same setting of Maskin and Riley (1983) in order to show that in some special cases the asymmetry increases the expected revenue in the FPA, unlike in the examples studied in Cantillon (2008).

[6] In order to circumvent this problem, some authors apply numerical methods: see for instance Fibich and Gavish (2011), Gayle and Richard (2008), Li and Riley (2007), and Marshall et al. (1994).

terms of the reverse hazard rate. The second assumption is more innovative and is related to a dispersive order among c.d.f.s, according to which the distribution of the strong bidder is more disperse than the distribution of the weak bidder; we refer to this assumption as to the "dispersion condition".[7] The approach in Kirkegaard (2011b) does not rely on differential equations, but on a well known result from mechanism design which establishes that the seller's expected revenue in an auction is given by the expected virtual valuation of the winner, at least when the bidders' lowest types have the same valuation (see Myerson, 1981). In the SPA the winner is the bidder with the highest valuation, but reverse hazard rate dominance and the dispersion condition imply that when the two bidders have the same valuation, the weak bidder has a higher virtual valuation than the strong bidder. Thus it is intuitive that the FPA is superior to the SPA if the weak bidder wins more often in the FPA than in the SPA. In fact, the property of reverse hazard rate dominance implies that in the FPA the weak bidder is more aggressive than the strong bidder, and therefore sometimes he wins even though his valuation is smaller than the valuation of the strong bidder. However, in some states of the world the weak bidder may be "too aggressive", and win even though his virtual valuation is smaller than the virtual valuation of the strong bidder. This makes the comparison between the FPA and the SPA not immediate, but Kirkegaard (2011b) shows that there is no ambiguity in expectation under the dispersion condition, as it implies that the expected virtual valuation of the winner (conditional on each given value of the weak bidder) is larger in the FPA than in the SPA.

As we mentioned above, some papers identify settings in which the seller prefers the SPA. For instance, Vickrey (1961) examines the case in which a bidder's valuation is common knowledge and the other bidder's value is uniformly distributed. The SPA dominates the FPA if the commonly known value is low enough. Maskin and Riley (2000a) consider the case in which a bidder's distribution is obtained from the other bidder's distribution by shifting some probability mass to the lower end-point, and in this case the SPA is superior if the initial distribution has an increasing hazard rate. In the binary setting we mentioned above, Maskin and Riley (1983) show that the SPA is better than the FPA if the bidders' high values are approximately equal, or if the probabilities of a high value are approximately equal.[8]

We compare the FPA with the SPA in the binary setting without the assumption that low values are equal, and find that for a broad set of parameters the SPA is superior to the FPA, as described above. Often, in order for the FPA to dominate the SPA it is necessary that the distribution of a bidder's valuation first order stochastically dominates the distribution of the other bidder's valuation "in a strong sense". For instance, for a not too large distribution shift we find that the SPA is superior to the FPA, unlike in Maskin and Riley (2000a).

[7] Clearly, these conditions are not necessary for the FPA to dominate the SPA. Lebrun (1996) and Cheng (2006) prove that the FPA is superior for some power distributions which violate the assumptions in Kierkegaard (2011b).

[8] Other specific cases in which SPA dominates FPA are found by Cheng (2010), in environments such that the equilibrium bidding functions for the FPA are linear, and by Gavious and Minchuk (2010), in examples such that the valuations' distributions are close to the uniform distribution.

2 The model

A (female) seller owns an indivisible object which is worthless to her and faces two (male) bidders. Let v_1 (v_2) denote the monetary valuation for the object of bidder 1 (bidder 2), which he privately observes; v_1 and v_2 are independently distributed. The set $\{v_{1L}, v_{1H}\}$ is the support for v_1, with $0 < v_{1L} < v_{1H}$ and $\lambda_1 \equiv \Pr\{v_1 = v_{1L}\} \in (0,1)$. Likewise, the support for v_2 is $\{v_{2L}, v_{2H}\}$ with $0 < v_{2L} < v_{2H}$ and $\lambda_2 \equiv \Pr\{v_2 = v_{2L}\} \in (0,1)$. Without loss of generality we assume that $v_{1L} \leq v_{2L}$. Both the seller and bidders are risk neutral, and a bidder's utility if he wins is given by his valuation for the object minus the price paid to the seller; his utility if he loses is zero. We use i_j to denote bidder i when his valuation is v_{ij}, thus for instance 2_L is the type of bidder 2 with valuation v_{2L}.

The main purpose of this paper is to evaluate the relative profitability of the FPA and the SPA for the seller. In either of these auctions each bidder submits simultaneously a nonnegative sealed bid, and the bidder who makes the highest bid wins the object (if the bidders tie, the winner is selected according to a specified tie-breaking rule: see next section). In the FPA the winning bidder pays the own bid; in the SPA he pays the loser's bid (i.e., the second highest bid).

3 Equilibrium bidding

3.1 SPA

It is well known that when bidders have private values, in the SPA it is weakly dominant for each bidder to bid the own valuation. Thus the seller's expected revenue R^S is the expectation of $\min\{v_1, v_2\}$, which is straightforward to evaluate (recall that $v_{1L} \leq v_{2L}$):

$$R^S = \begin{cases} \lambda_1 v_{1L} + (1-\lambda_1)(\lambda_2 v_{2L} + (1-\lambda_2)v_{2H}) & \text{if } v_{2H} \leq v_{1H} \\ \lambda_1 v_{1L} + (1-\lambda_1)(\lambda_2 v_{2L} + (1-\lambda_2)v_{1H}) & \text{if } v_{2L} \leq v_{1H} < v_{2H} \\ \lambda_1 v_{1L} + (1-\lambda_1)v_{1H} & \text{if } v_{1H} < v_{2L} \end{cases}. \tag{1}$$

For future reference, we denote with A the region of valuations such that $v_{1L} \leq v_{2L} < v_{2H} \leq v_{1H}$, with B the region such that $v_{1L} \leq v_{2L} \leq v_{1H} < v_{2H}$, and with C the region such that $v_{1L} < v_{1H} < v_{2L} < v_{2H}$. Therefore, (1) says that $R^S = \lambda_1 v_{1L} + (1-\lambda_1)(\lambda_2 v_{2L} + (1-\lambda_2)v_{2H})$ in region A; $R^S = \lambda_1 v_{1L} + (1-\lambda_1)(\lambda_2 v_{2L} + (1-\lambda_2)v_{1H})$ in region B; $R^S = \lambda_1 v_{1L} + (1-\lambda_1)v_{1H}$ in C. Notice that R^S does not depend on v_{2H} in regions B and C, and does not depend on v_{2L} in region C.

3.2 FPA

The analysis for the FPA is less immediate than for the SPA. In fact, finding the closed form for the equilibrium bidding strategies for an FPA with asymmetrically distributed valuations is often impossible when valuations are continuously distributed. However, this is not the case given our assumptions on the distributions of v_1 and v_2 (we consider equilibria in which no type of bidder bids above the own valuation). We typically find a mixed-strategy Bayes-Nash Equilibrium, but before describing it we consider a benchmark symmetric environment.

3.2.1 The benchmark symmetric setting

Suppose that v_1 and v_2 are symmetrically distributed such that $v_{1L} = v_{2L} \equiv v_L$, $v_{1H} = v_{2H} \equiv v_H$ and $\lambda_1 = \lambda_2 \equiv \lambda$. We know from Maskin and Riley (1985) that in this case the FPA has a unique Bayes-Nash Equilibrium and it is such that types 1_L and 2_L both bid v_L; types 1_H and 2_H play the same atomless mixed strategy with support $[v_L, \lambda v_L + (1-\lambda)v_H]$ and c.d.f. $G_H(b) = \frac{\lambda}{1-\lambda}\frac{b-v_L}{v_H-b}$. This implies that the object is efficiently allocated (i.e., in each state of the world the highest valuation bidder wins). Therefore, the expected revenue R^F in the FPA is equal to the expected social surplus $\lambda^2 v_L + (1-\lambda^2)v_H$ minus the bidders' aggregate rents $2[\lambda \cdot 0 + (1-\lambda)(v_H - \lambda v_L - (1-\lambda)v_H)]$, that is $R^F = (2\lambda - \lambda^2)v_L + (1-\lambda)^2 v_H$ (which is also equal to R^S).

3.2.2 The equilibrium for the asymmetric setting

For the setting with asymmetrically distributed v_1, v_2 described by Section 2, we find that often no pure-strategy Bayes-Nash equilibrium exists [the exception occurs when condition (3) below is satisfied], and sometimes no mixed-strategy Bayes-Nash equilibrium (BNE in the following) exists either. Precisely, when $v_{1L} = v_{2L}$ we find that no BNE exists in the standard FPA in which each bidder wins with probability $\frac{1}{2}$ in case of tie (for more details see below in this subsection and the proof to Proposition 1 in Section 6). However, Proposition 2 in Maskin and Riley (2000b) establishes that a BNE exists under a suitable tie-breaking rule such that each bidder i is required to submit both an "ordinary" bid $b_i \geq 0$ and a "tie-breaker" bid $c_i \geq 0$.[9] If $b_1 \neq b_2$, then c_1, c_2 are irrelevant but if $b_1 = b_2$ then bidder i wins if $c_i > c_j$ and pays $b_i + c_j$ (each bidder wins with probability $\frac{1}{2}$ if $b_1 = b_2$ and $c_1 = c_2$). Therefore c_1, c_2 are bids in a second price/Vickrey auction which takes place if and only if $b_1 = b_2$. In Proposition 1 we consider the FPA with this "Vickrey tie-breaking rule".

We want to stress that this particular tie-breaking rule is needed only when $v_{1L} = v_{2L}$, since existence is obtained for *any tie-breaking rule* if $v_{1L} \neq v_{2L}$. Precisely, when $v_{1L} < v_{2L}$ we find that multiple BNE exist regardless of the tie-breaking rule, but they are all outcome-equivalent. In particular, multiple BNE arise because type 1_L (and type 1_H in one case) never wins and needs to bid weakly less than v_{1L} (weakly less than v_{1H}) with probability one, in such a way that no type of bidder 2 has incentive to bid below v_{1L} (below v_{1H}). Since there are many strategies of 1_L (of 1_H) which achieve this goal,[10] multiple BNE exist. However, this multiplicity is only related to bids which are never winning bids and therefore, as we specified above, each BNE generates the same outcome in the sense that the allocation of the object, the payoff of each type of bidder and the expected revenue are the same; therefore multiplicity is not an issue.

[9] A very similar idea appears in Lebrun (2002), in the auction he denotes with $F\bar{P}A$.

[10] One example is such that 1_L bids according to the uniform distribution on $[\alpha v_{1L}, v_{1L}]$ with $\alpha < 1$ and close to 1.

Conversely, when $v_{1L} = v_{2L}$ in each BNE both types 1_L and 2_L bid v_{1L}, and (generically) also 1_H or 2_H bids v_{1L} with positive probability; suppose 2_H does so (to fix the ideas). Then 2_H ties with positive probability with 1_L by bidding v_{1L}, and if 2_H does not win the tie-break with probability one, he has an incentive to bid slightly above v_{1L}, which breaks the BNE. On the other hand, under the Vickrey tie-breaking rule, for a bidder i with valuation v_i submitting an ordinary bid b_i, it is weakly dominant to choose $c_i = v_i - b_i$, and in particular $c_{1L} = 0$, $c_{2H} = v_{2H} - v_{1L} > 0$ for the case we are considering; thus 2_H wins the tie-break paying v_{1L} in aggregate.[11] Given this property on weak dominance for tie-breaking bids, when we describe a strategy of bidder i we implicitly assume that to each ordinary bid b_i is associated a tie-breaking bid c_i equal to $v_i - b_i$. Therefore, whenever a tie occurs the bidder with the highest valuation wins and pays the valuation of the other bidder.

In the BNE described by Proposition 1(ii) below an important role is played by two specific bids $\hat{b}$ and $\bar{b}$ such that $\hat{b}$ is the smaller solution to the following quadratic equation (in the unknown b):

$$\lambda_2 b^2 + ((1-\lambda_2)v_{1H} + (\lambda_1 - \lambda_2)\, v_{2L} - \lambda_1 v_{1L} - v_{2H})b + ((1-\lambda_1)v_{2H} - (1-\lambda_2)v_{1H})v_{2L} + \lambda_1 v_{1L} v_{2H} = 0 \quad (2)$$

and $\bar{b} \equiv \lambda_2 \hat{b} + (1-\lambda_2)v_{1H}$. Precisely, $\hat{b}$ is the highest bid in the support of the mixed strategy of type 2_L, and $\bar{b}$ is the highest bid in the support of the mixed strategies of types 1_H and 2_H. The values of $\hat{b}$ and $\bar{b}$ are determined in such a way that the bidders' mixed strategies have no mass point at bids larger than v_{1L}, a necessary condition for equilibrium. The assumption (4) in Proposition 1(ii) implies that $\hat{b}$ satisfies $v_{1L} \le \hat{b} < \min\{v_{2L}, v_{1H}\}$.[12]

Proposition 1 *Given $v_{1L} \le v_{2L}$, consider the FPA with the Vickrey tie-breaking rule. Although multiple BNE may exist, they are all outcome-equivalent to the following BNE.*
Type 1_L always bids v_{1L} and the bids of the other types depend on the parameters as follows:
(i) If

$$v_{1H} \le \lambda_1 v_{1L} + (1-\lambda_1)v_{2L} \quad (3)$$

then types $2_L, 2_H$ bid v_{1H}; type 1_H bids weakly less than v_{1H} with probability one and in such a way that no type of bidder 2 has incentive to bid below v_{1H}.[13]
(ii) If

$$\lambda_1 v_{1L} + (1-\lambda_1)v_{2L} < v_{1H} < \frac{(1-\lambda_1)v_{2H} + (\lambda_1 - \lambda_2)v_{1L}}{1-\lambda_2} \quad (4)$$

then types $1_H, 2_L, 2_H$ play mixed strategies with support $[v_{1L}, \bar{b}]$ for 1_H, $[v_{1L}, \hat{b}]$ for 2_L, $[\hat{b}, \bar{b}]$ for 2_H, in which $\hat{b}$ is the smaller solution to (2) and $\bar{b} \equiv \lambda_2 \hat{b} + (1-\lambda_2)v_{1H}$. The c.d.f.s for the mixed

[11] In fact, whenever 1_L bids v_{1L} and ties with positive probability with type 2_j such that $v_{2j} > v_{1L}$, in each BNE 1_L selects $c_{1L} = 0$, otherwise it is profitable for 2_j to bid slightly above v_{1L}.

[12] See the proof of Proposition 1(ii).

[13] For instance, 1_H bids according to the uniform distribution on $[\alpha v_{1H}, v_{1H}]$ with $\alpha < 1$ and close to 1.

strategies of $1_H, 2_L, 2_H$ *are, respectively:*[14]

$$G_{1H}(b) = \begin{cases} \frac{\lambda_1(b-v_{1L})}{(1-\lambda_1)(v_{2L}-b)} & \textit{for } b \in [v_{1L}, \hat{b}] \\ \frac{1}{1-\lambda_1}(\frac{v_{2H}-\hat{b}}{v_{2H}-b} - \lambda_1) & \textit{for } b \in (\hat{b}, \bar{b}] \end{cases} \tag{5}$$

$$G_{2L}(b) = \frac{v_{1H} - \bar{b}}{\lambda_2(v_{1H} - b)}, \qquad G_{2H}(b) = \frac{1}{1-\lambda_2}(\frac{v_{1H} - \bar{b}}{v_{1H} - b} - \lambda_2). \tag{6}$$

(iii) If

$$\frac{(1-\lambda_1)v_{2H} + (\lambda_1 - \lambda_2)v_{1L}}{1-\lambda_2} \leq v_{1H} \tag{7}$$

then 2_L *bids* v_{1L} *and* $1_H, 2_H$ *play mixed strategies with a common support* $[v_{1L}, \lambda_1 v_{1L} + (1-\lambda_1)v_{2H}]$ *and the following c.d.f.s, for* $1_H, 2_H$ *respectively:*

$$G_{1H}(b) = \frac{\lambda_1}{1-\lambda_1}\frac{b - v_{1L}}{v_{2H} - b}, \qquad G_{2H}(b) = \frac{1}{1-\lambda_2}(\frac{v_{1H} - \lambda_1 v_{1L} - (1-\lambda_1)v_{2H}}{v_{1H} - b} - \lambda_2). \tag{8}$$

We discuss separately the three results in Proposition 1.

Case (i) When (3) holds, Proposition 1(i) establishes that each type of bidder 2 bids v_{1H} and wins for sure.[15] This occurs because v_{2L} is sufficiently larger than v_{1H}, which implies that each type of bidder 2 has so much to gain from winning that it is profitable for him to make a bid of v_{1H} in order to outbid each type of bidder 1. Precisely, (3) guarantees that type 2_L (and thus type 2_H as well) prefers winning for sure by bidding v_{1H} rather than bidding v_{1L} and winning only when facing type 1_L, that is with probability λ_1.

Case (ii) In the opposite case in which v_{1H} is large, (3) is violated and 2_L is not very aggressive since he prefers to bid v_{1L} and win only against 1_L rather than bidding v_{1H} and winning against both 1_L and 1_H (i.e., with certainty), as the latter alternative is too expensive. Indeed, 2_L bids in the interval $[v_{1L}, \hat{b}]$, with $\hat{b} < v_{1H}$, and with an atom at the bid $b = v_{1L}$, since $G_{2L}(v_{1L}) = \frac{v_{1H}-\bar{b}}{\lambda_2(v_{1H}-v_{1L})} > 0$. This less aggressive bidding of 2_L allows 1_H to win with positive probability by bidding in $(v_{1L}, \hat{b}]$, which makes his equilibrium payoff positive. This implies that the highest bid of 1_H is smaller than v_{1H}, since each bid in the support of a bidder's mixed strategy needs to maximize the expected payoff of the bidder given the strategies of the other types. Therefore also the highest bid of 2_H is smaller than v_{1H}, as we see from Proposition 1(ii). As v_{1H} increases, 2_L becomes increasingly less aggressive: $\hat{b}$ decreases and $G_{2L}(b)$ increases for any $b \in [v_{1L}, \hat{b})$. This occurs because as v_{1H} increases, the equilibrium payoff of 1_H increases and in order to satisfy the

[14] In the case that $\hat{b} = v_{1L}$ (which occurs if and only if $v_{1L} = v_{2L}$), 2_L bids v_{1L} and $\bar{b} = \lambda_2 v_{1L} + (1-\lambda_2)v_{1H}$, thus $G_{1H}(b) = \frac{1}{1-\lambda_1}(\frac{v_{2H}-\bar{b}}{v_{2H}-b} - \lambda_1)$ and $G_{2H}(b) = \frac{1}{1-\lambda_2}(\frac{v_{1H}-\bar{b}}{v_{1H}-b} - \lambda_2)$ for each $b \in [v_{1L}, \bar{b}]$.

[15] In a setting with continuously distributed valuations, Maskin and Riley (2000a) identify an analogous BNE and provide the intuition we describe here and immediately after Proposition 2. In addition, Maskin and Riley (1983) identify the BNE we describe in Proposition 1 for the case of $v_{1L} = v_{2L} = 0$. Thus our Proposition 1 is a new result for the case in which $v_{1L} < v_{2L}$ and (3) is violated.

condition of constant payoff of 1_H for bids in $(v_{1L}, \bar{b}]$ it is necessary that G_{2L} puts more weight on v_{1L} and becomes flatter in $(v_{1L}, \bar{b}]$.[16]

Case (iii) When v_{1H} is large enough such that (7) is satisfied, type 2_L bids v_{1L} with certainty and 2_H bids v_{1L} with positive probability. In particular, the larger is v_{1H}, the less aggressive 2_H becomes, giving higher probability to bids close to v_{1L}. We remark that (7) holds for a large λ_1, and thus for a large λ_1 type 2_L bids v_{1L} with probability one, type 2_H bids v_{1L} with positive probability. This occurs because a large λ_1 gives an incentive to bidder 2 to bid $b = v_{1L}$, as this (low) bid allows him to win against type 1_L, which arises with probability λ_1. Finally, notice that when (7) holds, the equilibrium strategies – and thus the expected revenue – do not depend on v_{2L}.

A well known feature of the FPA when valuations are asymmetrically distributed is that an inefficient allocation of the object occurs with positive probability. In our setting, suppose for instance that $v_{1L} < v_{2L} \neq v_{1H}$ and (4) holds. Then $\hat{b} > v_{1L}$ and in the state of the world with types $1_H, 2_L$ each type wins with positive probability; thus the highest valuation bidder may not win.

4 Comparison between the FPA and the SPA

In order to derive the seller's preferences between the FPA and the SPA we need to evaluate the expected revenue R^F in the FPA generated by the BNE described in Proposition 1. Although we can express R^F in closed form (see Subsection 6.3 in the appendix), the inefficiency of the FPA we mentioned above makes R^F a complicated function of the parameters, except when (3) is satisfied (in fact, in such a case the object is allocated efficiently). Under inequality (3), the comparison between R^F and R^S is straightforward, but when (3) is violated it is more difficult to obtain insights on the sign of $R^F - R^S$. Therefore we first examine the relatively simple case such that $\lambda_1 = \lambda_2$, and then we move to a more general setting without the assumption $\lambda_1 = \lambda_2$.

4.1 The case in which (3) is satisfied

When (3) holds we obtain a simple result, as described by next proposition.

Proposition 2 *If (3) is satisfied, then* $R^F > R^S$.

Proposition 2 is very simple to prove and to interpret. Precisely, (i) $R^F = v_{1H}$ when (3) is satisfied as both types of bidder 2 win the auction with a bid of v_{1H}; (ii) inequality (3) implies $v_{1H} < v_{2L}$ and thus from (1) we obtain $R^S = \lambda_1 v_{1L} + (1 - \lambda_1) v_{1H}$; (iii) since $v_{1L} < v_{1H}$, it follows that $R^F > R^S$. The intuition is that in both auctions bidder 2 always wins, thus R^S is equal to the expected valuation of the loser, bidder 1, but R^F is the high valuation of bidder 1. Notice that any profile of valuations which satisfies (3) belongs to region C.

[16] We describe a similar effect (with more details) in the intuition regarding Lemma 1 below.

4.2 The case in which $\lambda_1 = \lambda_2$

Under the assumption $\lambda_1 = \lambda_2$ we find the following interesting result.

Lemma 1 *Suppose that* $v_{1L} = v_{2L} = v_L$, $v_{1H} \neq v_{2H} = v_H$ *and* $\lambda_1 = \lambda_2 = \lambda$. *Then* R^F *is increasing in* v_{1H} *for* $v_{1H} \in (v_L, v_H]$ *and is decreasing in* v_{1H} *for* $v_{1H} \in [v_H, +\infty)$.

This lemma says that in a setting which is asymmetric only because $v_{1H} \neq v_H$, R^F is maximized with respect to v_{1H} at $v_{1H} = v_H$,[17] and in particular increasing v_{1H} above v_H reduces R^F.[18] In fact, it is somewhat surprising that, starting from a symmetric setting, an increase in the valuation of type 1_H generates a decrease in R^F. It seems reasonable to expect that an increase in v_{1H} above $v_H = v_{2H}$ makes type 1_H more aggressive than type 2_H, in the sense that 1_H bids (stochastically) higher than 2_H, and this occurs indeed in equilibrium. Crucially, however, it is not that 1_H bids more aggressively with respect to the symmetric setting, but rather type 2_H bids less aggressively. More in detail, notice that given $\lambda_1 = \lambda_2$, (7) is satisfied when $v_{1H} > v_H$ and therefore Proposition 1(iii) applies. This reveals that the behavior of types $1_L, 1_H, 2_L$ is unchanged with respect to the benchmark symmetric setting of Subsection 3.2.1: 1_L and 2_L both bid v_L, and 1_H plays a mixed strategy with support $[v_L, \lambda v_L + (1-\lambda)v_H]$ and c.d.f. $G_H(b) = \frac{\lambda}{1-\lambda}\frac{b-v_L}{v_H-b}$. On the other hand, now 2_H bids less aggressively than under the symmetric setting. Precisely, G_H and G_{2H} have the same support $[v_L, \lambda v_L + (1-\lambda)v_H]$, but since $G_{2H}(b) = \frac{(1-\lambda)(v_{1H}-v_H)+\lambda(b-v_L)}{(1-\lambda)(v_{1H}-b)}$ it is simple to verify that $G_{2H}(b) > G_H(b)$ for any $b \in [v_L, \lambda v_L + (1-\lambda)v_H)$, and in particular $G_{2H}(v_L) > 0 = G_H(v_L)$. Since 2_H is less aggressive with respect to the symmetric setting, it follows that an increase in v_{1H} has a negative effect on R^F. In fact, the larger is v_{1H} the higher (lower) is the probability that G_{2H} attaches to low (high) bids in $[v_L, \lambda v_L + (1-\lambda)v_H]$. As a consequence, R^F is monotonically decreasing with respect to v_{1H} for $v_{1H} > v_H$.

Naturally, this raises the question of why 2_H is less aggressive than in the symmetric setting. Suppose for a moment that 2_H still bids according to G_H even though $v_{1H} > v_H$. Then the payoff of type 1_H from bidding $b \in [v_L, \lambda v_L + (1-\lambda)v_H]$ is $(v_{1H} - b)[\lambda + (1-\lambda)G_H(b)]$. This is obviously higher than $(v_H - b)[\lambda + (1-\lambda)G_H(b)]$, his payoff before the increase in v_{1H}, and – more importantly – is increasing in b because the higher is b, the more likely is that 1_H wins and thus benefits from his higher valuation. In order to make 1_H indifferent among the bids in an interval $(v_L, b^*]$, with $b^* > v_L$, it is necessary that G_{2H} is flatter than G_H, and indeed $G_{2H}(b) = \frac{(1-\lambda)(v_{1H}-v_H)+\lambda(b-v_L)}{(1-\lambda)(v_{1H}-b)}$ has an atom at $b = v_L$ and grows more slowly than G_H for $b > v_L$. This is how an increase in v_{1H} generates a less aggressive behavior of 2_H. However, notice that the support for the mixed strategy of 2_H is still $[v_L, \lambda v_L + (1-\lambda)v_H]$, which requires that type 1_H still bids like in the symmetric setting in order to make 2_H indifferent among all the bids in $[v_L, \lambda v_L + (1-\lambda)v_H]$.[19]

[17] This fact may appear similar to the main message in Cantillon (2008), but in fact in our analysis the benchmark symmetric setting is fixed, whereas in Cantillon (2008) it is not.

[18] Obviously, an analogous result holds if v_{1H} is kept fixed and v_{2H} is allowed to vary.

[19] Lebrun (1998) considers a setting with continuously distributed valuations and assumes that the valuation distribution of one bidder changes into a new distribution which dominates the previous one in the sense of reverse

Lemma 1 suggests a simple result. Suppose that we start from the benchmark symmetric setting and let R^{F*} denote the resulting expected revenue. Then suppose that the valuation of 1_H is increased; this reduces the revenue below R^{F*} by Lemma 1. Finally, increase slightly the valuations of 1_L,2_L,2_H. Since R^F is a continuous function of the parameters, we infer that R^F remains smaller than R^{F*}, although the valuation of each type has increased with respect to the symmetric setting.

Proposition 3 *Consider the symmetric setting described in Subsection 3.2.1. Then, by suitably increasing the valuation of each type (but not each valuation by the same amount) we obtain a setting in which the revenue from the FPA is reduced.*

An instance in which the result in this proposition is obtained is such that $v_{1L} = v_{2L} = 100$, $v_{1H} = v_{2H} = 200$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$; then $R^{F*} = 125$. However, if $v_{1L} = v_{2L} = 105$, $v_{1H} = 400$ and $v_{2H} = 205$, then $R^F \simeq 123.12$.

Next proposition describes a set of circumstances which imply $R^S > R^F$ given $\lambda_1 = \lambda_2$. In doing so, it relies on Lemma 1 and on the fact that the BNE described by Proposition 1(iii) is independent of v_{2L}, for $v_{2L} \in [v_{1L}, v_{2H})$. The rest of this subsection is devoted to discussions and intuitions for these results.

Proposition 4 *Suppose that* $\lambda_1 = \lambda_2 \equiv \lambda$.
(i) $R^S > R^F$ *if at least one of the following conditions is satisfied:*

$$v_{1L} = v_{2L} \quad \text{and} \quad v_{1H} \neq v_{2H}; \tag{9}$$

$$v_{1L} < v_{2L} < v_{2H} \leq v_{1H}; \tag{10}$$

$$v_{1L} \leq v_{2L} \leq v_{1H} < v_{2H} \quad \text{with} \quad v_{2L} \text{ close to } v_{1L}; \tag{11}$$

$$v_{1L} < v_{2L} < v_{2H} \quad \text{with} \quad v_{2L} \leq v_{1H} + \frac{2\lambda - 1}{3 - 2\lambda}(v_{1H} - v_{1L}) \quad \text{and} \quad \lambda \geq \frac{1}{2}. \tag{12}$$

(ii) For values such that $v_{1L} < v_{1H} < v_{2L} < v_{2H}$*, the difference* $R^F - R^S$ *is increasing in* v_{2L}.

In terms of the regions A, B, C introduced in Subsection 3.1, Proposition 4(i) [condition (10)] reveals that $R^S > R^F$ in region A. The inequality $R^S > R^F$ holds also in region B for v_{2L} close to v_{1L}, and in the whole region B if $\lambda \geq \frac{1}{2}$: see conditions (11) and (12).[20] Figure 1 in Subsection 4.2.3 provides a graphical representation of these results for the case of $\lambda \geq \frac{1}{2}$.

Finally, Proposition 4(ii) establishes that in region C, $R^F - R^S$ is increasing with respect to v_{2L}, that is an increase in v_{2L} favors the FPA with respect to the SPA. This is consistent with Proposition 2, since an increase in v_{2L} brings us closer to satisfying (3), which implies $R^F > R^S$.

hazard rate domination (the support is unchanged). He show that, as a consequence, for each bidder the new bid distribution first order stochastically dominates the initial bid distribution, and thus the expected revenue increases.

[20] In particular, the SPA is better than the FPA for any small deviation from the symmetric setting, that is when $v_{2L} - v_{1L}$ and $v_{2H} - v_{1H}$ are close to zero, but $v_{2L} - v_{1L} > 0$ and/or $v_{2H} - v_{1H} \neq 0$.

4.2.1 Condition (9): $v_{1L} = v_{2L}$ and $v_{1H} \neq v_{2H}$

We start by considering (9), and suppose that $v_{1H} > v_{2H}$. Then from Lemma 1 we deduce that $R^S > R^F$ since an increase in v_{1H} above v_{2H} reduces R^F but does not affect the distribution of $\min\{v_1, v_2\}$, and thus R^S does not change.

For the case of $v_{1H} < v_{2H}$, consider the symmetric setting with low valuations both equal to $v_{1L} = v_{2L}$ and high valuations both equal to v_{1H}; then $R^F = R^S$. Now increase the valuation of type 2_H above v_{1H} to obtain the asymmetric setting we are considering. Although R^S does not change, the logic of Lemma 1 (see footnote 18) reveals that R^F decreases. Hence $R^F < R^S$.

We have thus established that (9) implies $R^S > R^F$ as a corollary of Lemma 1, but we notice that Maskin and Riley (1985) (in their Section III) consider the setting of Proposition 4, except that they assume $v_{1L} = v_{2L} = 0$, and claim that an increase in v_{2H} above v_{1H} favors the FPA over the SPA, in contrast with Proposition 4. However, they do not provide a formal proof of their claim. On the other hand, Maskin and Riley (1983) conclude that $R^S > R^F$, consistently with Proposition 4(i): see their Figure 1 between pages 18 and 19.[21]

4.2.2 Condition (10): $v_{1L} < v_{2L} < v_{2H} \leq v_{1H}$

Condition (10) has effects which are almost straightforward. In case that $v_{2H} = v_{1H}$, (7) is satisfied and Proposition 1(iii) applies. Hence R^F is equal to the revenue in the symmetric setting with both low valuations equal to v_{1L} since (as we mentioned in Subsection 3.2.2) R^F does not depend on $v_{2L} \in (v_{1L}, v_{2H})$. However, (1) reveals that R^S is increasing in v_{2L} and therefore $R^S > R^F$.

In case that $v_{2H} < v_{1H}$, suppose first that $v_{1L} = v_{2L}$. We know from condition (9) that $v_{2H} < v_{1H}$ implies $R^S > R^F$, and the previous paragraph explains that an increase in v_{2L} has no effect on R^F, but increases R^S. Hence $R^S > R^F$ still holds.

$\mathbf{v}_1$ more uncertain than $\mathbf{v}_2$ The inequalities in (10) characterize the setting in which v_1 has a wider range of variability than v_2; this includes the special case in which v_1 is a mean-preserving-spread of v_2. In this setting the ranking between R^S and R^F is unambiguous: the SPA is better than the FPA when a bidder's valuation is more uncertain then the other bidder's valuation.

Kirkegaard (2011a) notices that only Vickrey (1961) provides a theoretical ranking result without assuming first order stochastic dominance between the bidders' distributions of valuations.[22] Precisely, Vickrey (1961) assumes that v_1 is uniformly distributed over $[0, 1]$ and v_2 is common knowledge, equal to a fixed value a; he proves that the FPA is superior to the SPA for $a > 0.43$. Now consider in our framework the parameters $\lambda = \frac{1}{2}$ and $v_{1L} = 0$, $v_{1H} = 1$, $v_{2L} = a - \varepsilon$, $v_{2H} = a + \varepsilon$

[21] Since they assume $v_{1L} = v_{2L} = 0$, Maskin and Riley (1983) do not consider the various cases covered in Proposition 4, and they do not have the results in Lemma 1 and Proposition 5.

[22] Gayle and Richard (2008), Li and Riley (1999) and Li and Riley (2007) apply numeric analysis to settings without first order stochastic dominance and obtain mixed results.

with $\varepsilon > 0$ and close to zero.[23] This setting is in a sense similar to that in Vickrey (1961) since v_1 is uniformly distributed over $\{0, 1\}$, and v_2 is almost commonly known to be equal to a.[24] However, Proposition 4(i) [condition (10)] establishes that $R^S > R^F$ for any $a \in (0, 1)$. This difference with respect to Vickrey (1961) arises because in our setting R^F is considerably lower than in Vickrey (1961), due to the fact that type 2_L bids $v_{1L} = 0$ with certainty (and type 2_H bids 0 with positive probability), as bidding 0 suffices to win the auction if his opponent is type 1_L, an event with probability $\frac{1}{2}$. It is this incentive of bidder 2 to play "low-ball" that makes R^F small.[25] Conversely, no such effect appears when v_1 is uniformly distributed over $[0, 1]$ because if bidder 2 bids close to zero then he wins only against a small set of types of bidder 1. For instance, if $a = \frac{1}{2}$ then Vickrey (1961) proves that bidder 2's equilibrium mixed strategy has support $[\frac{1}{4}, \frac{7}{16}]$, that is 2's minimum bid is $\frac{1}{4}$.

4.2.3 Conditions (11) and (12)

Given the innocuous assumption that $v_{1L} \leq v_{2L}$, after (9) and (10) have been considered, the only class of asymmetry remaining given $\lambda_1 = \lambda_2$ is such that $v_{1L} < v_{2L}$ and $v_{1H} < v_{2H}$, which implies that the distribution of v_2 first order stochastically dominates the distribution of v_1. In this setting, (11) establishes that $R^S > R^F$ when $v_{2L} - v_{1L}$ is close to zero, a consequence of (9). But in fact, if $\lambda \geq \frac{1}{2}$ then $R^S > R^F$ holds even though $v_{2L} - v_{1L}$ is not small, as long as $v_{2L} \leq v_{1H}$, that is in the whole region B. In words, in order for $R^F > R^S$ to hold it is not sufficient that the distribution of v_2 first order stochastically dominates the distribution of v_1, but it is necessary that v_{2L} is sufficiently larger than v_{1L}; when $\lambda \geq \frac{1}{2}$, $R^F > R^S$ actually requires $v_{2L} > v_{1H}$, which means that the profile of valuations is in region C. In this region an increase in v_{2L} favors the FPA with respect to the SPA, which is consistent with Proposition 2 as we noticed above.

It is interesting to inquire why a higher value of λ enlarges the set of valuations for which we can prove that $R^S > R^F$ and, in short, the reason is that a larger λ makes the bidders less aggressive in the FPA (but obviously does not affect their behavior in the SPA). In order to explain how (12) is obtained, recall that in our final remark in Subsection 3.2.2 we noticed that in the BNE described by Proposition 1(ii) the highest valuation bidder does not always win. Conversely, the efficient allocation is always achieved in the SPA. Therefore a sufficient condition for $R^S > R^F$ is that the aggregate bidders' rents in the FPA, U^F, are (weakly) larger than the rents in the SPA, U^S.[26] It turns out that $U^F \geq U^S$ reduces to $v_{1L} + v_{2L} \geq 2\hat{b}$ when the valuations are in region B, and to $v_{1L} + 2v_{1H} - v_{2L} \geq 2\hat{b}$ when the valuations are in region C. This suggests that the SPA is more likely to be superior to the FPA the smaller is $\hat{b}$, which is quite intuitive as $\hat{b}$ can be viewed as an index of how bidders are aggressive in the FPA, given that the highest bids submitted by types

[23] Proposition 1 still holds even though $v_{1L} = 0$ violates our assumption $v_{1L} > 0$. However, when $v_{1L} = 0$ the Vickrey tie-breaking rule is needed also if $v_{1L} \neq v_{2L}$.

[24] If we set $\varepsilon = 0$, then $v_{2L} = v_{2H}$, which violates the assumption $v_{2L} < v_{2H}$, but nevertheless $\lambda G_{2L}(b) + (1 - \lambda)G_{2H}(b)$ is the c.d.f. of the equilibrium mixed strategy of bidder 2 when $v_{2L} = v_{2H}$.

[25] This effect appears also in Example 3 in Maskin and Riley (2000a).

[26] Each of conditions (11) and (12) guarantees indeed that $U^F \geq U^S$.

$1_H, 2_L, 2_H$ are $\bar{b}, \hat{b}, \bar{b}$, respectively, with $\bar{b} = \lambda\hat{b} + (1-\lambda)v_{1H}$. In order to inquire how $\hat{b}$ depends on λ, we need to recall that the support for the mixed strategy of type 2_L is $[v_{1L}, \hat{b}]$, and $\lambda(v_{2L} - v_{1L})$ is the rent of 2_L, the expected payoff he obtains by bidding $b = v_{1L}$. Also the bid $b = \hat{b}$ needs to yield 2_L the same payoff $\lambda(v_{2L} - v_{1L})$, and this suggests that $\hat{b}$ is decreasing in λ [indeed we can use (2) to prove this result formally].

In Figure 1 we fix $\lambda \geq \frac{1}{2}$ and v_{1L}, v_{1H}, and partition the space (v_{2L}, v_{2H}) in two regions S and F such that $R^S > R^F$ if $(v_{2L}, v_{2H}) \in S$, and $R^F \geq R^S$ if $(v_{2L}, v_{2H}) \in F$ – obviously, the feasible values of (v_{2L}, v_{2H}) are above the line $v_{2H} = v_{2L}$. In particular, $S_{(iii)}$ is region A, the set in which (10) is satisfied [in this case (7) holds and the BNE of Proposition 1(iii) applies]; $F_{(i)}$ is the set in which (3) holds [then the BNE of Proposition 1(i) applies]. The remaining set includes the whole region B and a subset of C, and is such that (4) is satisfied – thus the BNE of Proposition 1(ii) applies. The boundary between S and F is obtained numerically.

insert Figure 1 here

Caption Figure 1: Comparison between the FPA and the SPA when $\lambda_1 = \lambda_2 \geq \frac{1}{2}$. In the dark grey region $S = S_{(ii)} \cup S_{(iii)}$ the SPA dominates the FPA in terms of the seller's revenue. In the light grey region $F = F_{(i)} \cup F_{(ii)}$ the FPA is superior. Proposition 1(i) applies in the north-east set $F_{(i)}$, 1(ii) in the set $F_{(ii)} \cup S_{(ii)}$ in the middle and north-west, and 1(iii) in the south-west set $S_{(iii)}$.

Distribution shift and rescaling A particular type of asymmetry considered in the literature is as follows. Given the c.d.f. F_1 for the valuation of bidder 1, the c.d.f. for v_2 is $F_2(v_2) = F_1(v_2 - \alpha)$ with $\alpha > 0$, that is F_2 is obtained by shifting F_1 to the right, which implies that bidder 2 is ex ante stronger than 1. In a setting with continuously distributed values, Maskin and Riley (2000a) prove that under suitable assumptions on F_1 (which include convexity and log-concavity) the FPA generates a higher revenue than the SPA; Kirkegaard (2011b) obtains the same result under weaker assumptions. In our context this sort of asymmetry is obtained by fixing v_{1L}, v_{1H} and setting $v_{2L} = v_{1L} + \alpha$, $v_{2H} = v_{1H} + \alpha$, for some $\alpha > 0$. From (11) and (12) we can obtain sufficient conditions for $R^S > R^F$, but in fact in the appendix we exploit this particular structure of asymmetry to prove a stronger result: $R^S > R^F$ as long as $\frac{\alpha}{v_{1H} - v_{1L}} \leq \frac{2\lambda}{2-3\lambda}$ (for $\lambda \leq \frac{2}{5}$) or $\frac{\alpha}{v_{1H} - v_{1L}} \leq \frac{2(2+\lambda)}{3(2-\lambda)}$ (for $\lambda > \frac{2}{5}$).

These results have an immediate interpretation: In our discrete setting a small shift, that is a small $\alpha > 0$, favors the SPA over the FPA, whereas the result is reversed for a large shift.[27] On the other hand, in their numeric analysis applied to continuous distributions, Li and Riley (2007) find that a shift "can result in economically very significant revenue differences [in favor of the FPA]" for examples with uniform or truncated normal distributions, and claim that "Analysis of other distributions also produces broadly similar results". Our results show that this claim does not hold in a setting with binary supports.

[27] For instance, $R^F > R^S$ definitely holds if $\alpha > 0$ is such that (3) is satisfied, that is if $\frac{\alpha}{v_{1H} - v_{1L}} \geq \frac{1}{1-\lambda}$.

In fact, it is possible to see the result that $R^S > R^F$ for a small shift as a consequence of Lemma 1. Precisely, (i) for $\alpha = 0$ there is no shift and we are in the benchmark symmetric setting of Subsection 3.2.1; (ii) from (2) we find that $\left.\frac{\partial \hat{b}}{\partial \alpha}\right|_{\alpha=0} = 0$, thus $\left.\frac{\partial \bar{b}}{\partial \alpha}\right|_{\alpha=0} = 0$; (iii) for any $\alpha > 0$, (4) holds and thus (6) in Proposition 1(ii) reveals that a small $\alpha > 0$ generates a zero first order change in the bidding of types 2_L and 2_H; (iv) the logic of Lemma 1 [see footnote 18, or equivalently see (5)] reveals that 1_H bids less aggressively for a small $\alpha > 0$ than for $\alpha = 0$. Therefore a small shift reduces R^F but increases R^S, which implies $R^F < R^S$.

Example 4 in Kirkegaard (2011a) starts from F_2 such that $F_2(e^v)$ is convex and log-concave and obtains F_1 as $F_1(v) = F_2(\gamma v)$ for some $\gamma > 1$ and not too large; thus v_1 is a rescaling of v_2, and Kirkegaard (2011a) proves that $R^F > R^S$. In our context this sort of asymmetry is obtained by fixing v_{2L}, v_{2H} and setting $v_{1L} = \frac{1}{\gamma}v_{2L}$, $v_{1H} = \frac{1}{\gamma}v_{2H}$. The comparison between the SPA and the FPA yields results which are different from those in Kirkegaard (2011a), but are similar to the results obtained for a shift. Precisely, (11) reveals that $R^S > R^F$ if γ is not much larger than 1 (i.e., for a small rescaling), whereas a large γ makes (3) satisfied and thus $R^F > R^S$.

4.2.4 The distribution of bids in the FPA and the bidders' preferences

For $i = 1, 2$, let G_i denote the ex ante c.d.f. of the equilibrium bids submitted by bidder i in the FPA, that is $G_i(b) = \lambda G_{iL}(b) + (1-\lambda)G_{iH}(b)$. Using Proposition 1 we can compare the equilibrium bid distributions of bidder 1 and 2 in the FPA, and we find that G_2 first order stochastically dominates G_1 when $v_{2H} > v_{1H}$; the opposite result obtains if $v_{1H} > v_{2H}$. Notice that when $v_{2H} > v_{1H}$, the distribution of v_2 first order stochastically dominates the distribution of v_1 and the result that G_2 first order stochastically dominates G_1 agrees with Corollary 1 in Kirkegaard (2009), for a setting with continuous distributions. On the other hand, when $v_{2H} < v_{1H}$ there is no first order stochastic dominance between the distribution of v_1 and v_2, but second order stochastic dominance applies if $v_{1H} \leq v_{2H} + \frac{\lambda}{1-\lambda}(v_{2L} - v_{1L})$, that is if the expected value of v_2 is weakly larger than the expected value of v_1. Under second order stochastic dominance between the valuations distributions, Proposition 5 in Kirkegaard (2009) shows that the bid distributions must cross, whereas we find that G_1 first order stochastically dominates G_2.

Proposition 1 also allows us to compare the bidders' payoffs in the FPA with their payoffs in the SPA: it turns out that bidder 1 weakly prefers the FPA, whereas bidder 2 weakly prefers the SPA. These results largely agree with the results in Propositions 3.3(ii) and 3.6 in Maskin and Riley (2000a).

4.2.5 Relationship with Kirkegaard (2011b)

Proposition 4(i) reveals that $R^S > R^F$ for a broad set of deviations from the benchmark symmetric setting, provided that $\lambda_1 = \lambda_2$. On the other hand, a frequent result in the literature on asymmetric auctions is that $R^F > R^S$. Since the most general theoretical results are obtained in Kirkegaard (2011b), we explain why his analysis does not apply to our setting.

Kirkegaard (2011b) considers a two-bidder environment with supports $[\beta_1, \alpha_1]$ for v_1 and $[\beta_2, \alpha_2]$ for v_2 such that $\beta_1 \leq \beta_2$ and $\alpha_1 < \alpha_2$. The c.d.f.s F_1, F_2 have no atoms and have continuous and positive densities f_1, f_2 in the respective supports; moreover, 1 is ex ante weaker than 2 in the sense that F_2 first order stochastically dominates F_1. A crucial ingredient for the result is $r(v)$, which is defined as $F_2^{-1}[F_1(v)]$ for each $v \in [\beta_1, \alpha_1]$, that is $r(v)$ satisfies $\Pr\{v_2 \leq r(v)\} = \Pr\{v_1 \leq v\}$ and $r(v) \geq v$ as F_2 first order stochastically dominates F_1. The main result in Kirkegaard (2011b), Theorem 1, establishes that $R^F > R^S$ if[28]

$$\frac{f_2(v)}{F_2(v)} \geq \frac{f_1(v)}{F_1(v)} \quad \text{for any} \quad v \in [\beta_1, \alpha_1] \cap [\beta_2, \alpha_2]; \tag{13}$$

$$f_1(v) \geq f_2(x) \quad \text{for any} \quad x \in [v, r(v)] \quad \text{and} \quad \text{any} \quad v \in [\beta_1, \alpha_1]. \tag{14}$$

This theorem results from a clever application of the mechanism design techniques introduced by Myerson (1981), and precisely relies on the following argument. The expected revenue in either auction is given by the expected virtual valuation of the winning bidder minus the rents of the lowest types β_1 and β_2 of the two bidders. In the SPA bidder 1 wins if and only if $v_1 > v_2$. However, (13) and (14) imply that the virtual valuation of 1 is larger than the virtual valuation of 2 when valuations are equal, which suggests that it is profitable to have 1 winning the auction if $v_1 = v_2$, or if v_1 is slightly larger than v_2. In fact, (13) implies that in the FPA bidder 1 bids higher than 2 for equal valuations. Thus 1 wins when $v_2 < v_1$, and also when $v_2 < k^F(v_1)$ for a certain function k^F such that $v < k^F(v) \leq r(v)$ (the latter inequality means that the ex ante equilibrium bid distribution of 2 first order stochastically dominates the ex ante bid distribution of 1). This suggests that the FPA is more profitable than the SPA, but in fact in some states of the world bidder 1 may win even though his virtual valuation is smaller than the virtual valuation of 2. As a consequence, it is not obvious that the FPA dominates the SPA, but Kirkegaard (2011b) shows that if $\beta_1 = \beta_2$, then (14) implies that the expected virtual valuation of the winner, conditional on v_1, is larger in the FPA than in the SPA for each v_1. If instead $\beta_1 < \beta_2$, then the above result may not hold, but the FPA extracts from type β_2 of bidder 2 a higher rent than the SPA, which allows to prove that $R^F > R^S$.

The assumptions in Kirkegaard (2011b) obviously rule out our discrete setting, but given the c.d.f.s

$$\tilde{F}_1(v_1) = \begin{cases} 0 & \text{if} \quad v_1 < v_{1L} \\ \lambda & \text{if} \quad v_{1L} \leq v_1 < v_{1H} \\ 1 & \text{if} \quad v_{1H} \leq v_1 \end{cases}, \qquad \tilde{F}_2(v_2) = \begin{cases} 0 & \text{if} \quad v_2 < v_{2L} \\ \lambda & \text{if} \quad v_{2L} \leq v_2 < v_{2H} \\ 1 & \text{if} \quad v_{2H} \leq v_2 \end{cases}$$

[28] Condition (13) is a standard condition of dominance in terms of reverse hazard rates. On the other hand, (14) is innovative and Kirkegaard (2011a) proves that it implies that $r(v) - v$ is increasing, which means that F_2 is more disperse than F_1 according to a specific order of dispersion between c.d.f. Moreover, Kirkegaard (2011a) gives an economic interpretation to (14) linked to the relative steepness of the demand function of bidder 1 with respect to the demand function of bidder 2.

for v_1, v_2 in our model, we can approximate $\tilde{F}_1, \tilde{F}_2$ using atomless c.d.f.[29] Precisely, consider two sequences of atomless c.d.f. $\{F_1^n, F_2^n\}_{n=1}^{+\infty}$, with continuous and positive densities f_1^n, f_2^n for each n, which converges weakly to $\tilde{F}_1, \tilde{F}_2$. We prove in Section 10 that for any large n, (13) and/or (14) are violated by F_1^n, F_2^n.

4.3 The general case

In this subsection we remove the assumption $\lambda_1 = \lambda_2$. Our results for this case, described by Proposition 5 below, are less clear cut than when $\lambda_1 = \lambda_2$, but however they offer some insights on which format is likely to perform better in different settings.

Proposition 5 *(i) For any λ_1 and λ_2, suppose that $v_{1H} = v_{2H}$. Then $R^S > R^F$ holds as long as $v_{1L} < v_{2L}$ and/or $\lambda_1 \neq \lambda_2$.*

(ii) The case of $\lambda_2 \geq \lambda_1$.

(iia) $R^S > R^F$ in region B if v_{2L} is close to v_{1L}; $R^S > R^F$ in the whole region B if $\lambda_2 \geq \max\{\frac{1}{2}, \lambda_1\}$.

(iib) The difference $R^F - R^S$ is increasing with respect to v_{2L} in region C.

(iii) The case of $\lambda_1 \geq \lambda_2$.

(iiia) $R^S > R^F$ in region A.

(iiib) Suppose that $\lambda_1 \geq \lambda_2(1 + \ln \frac{1}{\lambda_2})$, and consider regions B and C. If $v_{2L} \leq v_{1H}$ or if v_{2L} is not too larger than v_{1H}, then there exists v_{2H}^* (and $v_{2H}^* > v_{1H}$) such that $R^S > R^F$ when $v_{2H} \in (v_{2L}, v_{2H}^*)$, but $R^F > R^S$ when $v_{2H} > v_{2H}^*$. If conversely v_{2L} is much larger than v_{1H}, then $R^F > R^S$ for any $v_{2H} > v_{2L}$.

Proposition 5(i) is in a sense quite intuitive, since we know that $R^S > R^F$ when $v_{1H} = v_{2H}$ if (i) $v_{1L} < v_{2L}$ and $\lambda_1 = \lambda_2$ [from Proposition 4(i), condition (10)], or (ii) $v_{1L} = v_{2L}$ and $\lambda_1 \neq \lambda_2$ (from Maskin and Riley, 1983). Proposition 5(i) essentially verifies that $R^S > R^F$ still holds if both inequalities $v_{1L} < v_{2L}$ and $\lambda_1 \neq \lambda_2$ hold.

A simple way to see why $R^S > R^F$ when $v_{1H} = v_{2H}$ consists in arguing as in Subsection 4.2.3, and proving that the bidders' rents are larger in the FPA than in the SPA. Precisely, when $v_{1H} = v_{2H}$ condition (3) is violated and (7) reduces to $\lambda_1 \geq \lambda_2$; therefore Proposition 1(iii) applies if $\lambda_1 \geq \lambda_2$, and Proposition 1(ii) applies if $\lambda_1 < \lambda_2$. In the proof to Proposition 5(i) we show that bidder 1 (bidder 2) strictly (weakly) prefers the FPA to the SPA since (i) 1_H earns zero in the SPA when facing 2_H, earns $v_{1H} - v_{2L}$ against 2_L; (ii) 1_H can beat 2_L in the FPA by bidding v_{1L} or $\hat{b}$ (depending on whether $\lambda_1 \geq \lambda_2$ or $\lambda_1 < \lambda_2$), and both v_{1L} and $\hat{b}$ are smaller than v_{2L}. Likewise, the payoff of bidder 2 in the SPA is zero against 1_H, is $v_2 - v_{1L}$ against 1_L. The FPA is certainly not worse for 2 as he can beat 1_L by bidding v_{1L}.[30]

[29] Lebrun (2002) establishes that the equilibrium correspondence is upper hemicontinuous with respect to the valuation distributions, for the weak topology. Given that all BNE are outcome-equivalent at each given information structure, it follows that the equilibrium correspondence is in fact continuous. Therefore also R^F is continuous, as it is the expectation of a continuous function of bids (the maximum).

[30] Here the bidders have the same preferences between the FPA and the SPA, whereas under the assumptions on Maskin and Riley (2000a) that is never the case.

Proposition 5(ii) considers the case of $\lambda_2 \geq \lambda_1$ and generalizes the results in Proposition 4(i) linked to conditions (11) and (12). Precisely, when $\lambda_2 > \lambda_1$ and $v_{1L} = v_{2L}$ we have that R^F decreases if v_{2H} increases above v_{1H}, as when $\lambda_2 = \lambda_1$; this makes R^F smaller than R^S for $v_{2H} > v_{1H}$ and v_{2L} close to v_{1L}. Regarding the inequality $R^S > R^F$ in the whole region B if $\lambda_2 \geq \max\{\frac{1}{2}, \lambda_1\}$, the intuition is that for a large λ_2, v_2 is almost commonly known to be equal to v_{2L} such that $v_{1L} < v_{2L} \leq v_{1H}$. In such a case, we know from subsection 4.2.2 that $R^S > R^F$ (see footnote 24), a result suggested also by Example 3 in Maskin and Riley (2000a). Hence, in regions B and C, a figure qualitatively similar to figure 1 applies for the case of $\lambda_2 \geq \max\{\frac{1}{2}, \lambda_1\}$,[31] and $R^F > R^S$ requires v_{2L} larger than v_{1H}.

On the other hand, the result in Proposition 4(i) related to condition (10) does not extend to the case of $\lambda_2 > \lambda_1$ because then $R^S > R^F$ fails to hold for some profile of valuations in region A. Precisely, consider $(v_{1L}, v_{1H}, v_{2L}, v_{2H})$ in region A with v_{2L} very close to v_{2H}, and suppose that (7) is satisfied with equality [(7) does not depend on v_{2L}]. In this case the valuation of bidder 2 is almost common knowledge, and then Proposition 4(i) [condition (10)] applies even though $\lambda_2 \neq \lambda_1$, since v_{2L} close to v_{2H} makes the precise value of λ_2 almost irrelevant; thus $R^S > R^F$. If now we consider a reduction of v_{2L} from about v_{2H} to about v_{1L}, then R^F is unaffected since (7) is still satisfied and the equilibrium bidding in the FPA does not depend on v_{2L}. On the other hand, the reduction in v_{2L} reduces R^S because the revenue in the SPA is equal to v_{2L} with probability $(1-\lambda_1)\lambda_2$ in region A. In particular, R^S is reduced considerably when λ_2 is large and λ_1 is small, consistently with $\lambda_2 > \lambda_1$. In such a case $R^S < R^F$ if v_{2L} is close to v_{1L}.

Conversely, Proposition 5(iiia) extends the result of Proposition 4(i) [condition (10)] for region A to the case of $\lambda_1 \geq \lambda_2$. The reason is that a reduction of λ_2 below λ_1 does not affect R^F, whereas it increases R^S [see (8) and (1)].

Proposition 5(iiib) considers regions B and C and shows that when λ_1 is large with respect to λ_2, $R^F > R^S$ if and only if v_{2H} is sufficiently large. In Figure 2 we fix λ_1, λ_2 such that $\lambda_1 \geq \lambda_2(1+\ln\frac{1}{\lambda_2})$, we fix v_{1L}, v_{1H}, and we partition the space (v_{2L}, v_{2H}) in two regions S and F such that $R^S > R^F$ if $(v_{2L}, v_{2H}) \in S$, and $R^F \geq R^S$ if $(v_{2L}, v_{2H}) \in F$ – obviously, the feasible values of (v_{2L}, v_{2H}) are above the line $v_{2H} = v_{2L}$. In particular, $F_{(i)}$ and $F_{(ii)}$ are the sets in which (3) and (4) are satisfied, respectively. The remaining set $S_{(iii)} \cup F_{(iii)}$ is such that (7) is satisfied. The boundary between S and F is obtained numerically.

insert Figure 2 here

Caption Figure 2: Comparison between the FPA and the SPA when $\lambda_1 \geq \lambda_2(1+\ln\frac{1}{\lambda_2})$. In the dark grey region $S = S_{(iii)}$ the SPA dominates the FPA in terms of the seller's revenue. In the light grey region $F = F_{(i)} \cup F_{(ii)} \cup F_{(iii)}$ the FPA is superior. Proposition 1(i) applies in the north-east set

[31] In fact, when $\lambda_2 \geq \max\{\frac{1}{2}, \lambda_1\}$ we prove in Section 11 that $R^S > R^F$ if $v_{1L} < v_{2L} < v_{2H}$ and $v_{2L} \leq v_{1H} + \frac{\lambda_1(3\lambda_2-\lambda_2^2-\lambda_1\lambda_2-1)}{\lambda_2(1-\lambda_1)(3-\lambda_1-\lambda_2)}(v_{1H}-v_{1L})$, with $3\lambda_2 - \lambda_2^2 - \lambda_1\lambda_2 - 1 > 0$ since $\lambda_2 \geq \max\{\frac{1}{2}, \lambda_1\}$. When $\lambda_1 = \lambda_2 = \lambda$, the inequality $v_{2L} \leq v_{1H} + \frac{\lambda_1(3\lambda_2-\lambda_2^2-\lambda_1\lambda_2-1)}{\lambda_2(1-\lambda_1)(3-\lambda_1-\lambda_2)}(v_{1H}-v_{1L})$ boils down to $v_{2L} \leq v_{1H} + \frac{2\lambda-1}{3-2\lambda}(v_{1H}-v_{1L})$ as in (12).

$F_{(i)}$, 1(ii) in the set $F_{(ii)}$ in the middle and north-west, and 1(iii) in the south-west set $F_{(iii)} \cup S_{(iii)}$.

Remarkably, this result is the opposite of the result obtained when λ_2 is large with respect to λ_1, as in such a case $R^S > R^F$ in the whole region B. In order to understand the source of this difference, suppose $v_{2H} = v_{1H}$; then we know that $R^S > R^F$ from Proposition 5(i). For a large λ_1, inequality (7) is satisfied and thus Proposition 1(iii) applies for the FPA, as we explained in Subsection 3.2.2. In this setting, increasing v_{2H} makes both types 1_H and 2_H more aggressive, which increases R^F. However, an increase in v_{2H} has no effect on R^S and $R^F > R^S$ holds if v_{2H} is sufficiently large such that (7) is violated.[32] Conversely, if λ_2 is large we find that an increase in v_{2H} above v_{1H} may increase or decrease R^F, depending on the other parameter values, but however $R^S > R^F$ since v_2 is almost common knowledge for a large λ_2, as mentioned above.

5 A setting with three types for each bidder

In this section we consider a setting in which the support for each bidder's valuation is a three-point set. Precisely, the set $\{v_{1L}, v_{1M}, v_{1H}\}$ is the support for v_1 and the set $\{v_{2L}, v_{2M}, v_{2H}\}$ is the support for v_2, with $v_{iL} < v_{iM} < v_{iH}$ and $\lambda_L \equiv \Pr\{v_i = v_{iL}\} > 0$, $\lambda_M \equiv \Pr\{v_i = v_{iM}\} > 0$, $\lambda_H \equiv \Pr\{v_i = v_{iH}\} > 0$ for $i = 1, 2$. We still use R^F (R^S) to denote the expected revenue under the FPA (under the SPA). As usual, R^S is the expectation of $\min\{v_1, v_2\}$.

In this environment we do not characterize a BNE for the FPA for all parameters values, but nevertheless we can prove that some of the results described in Subsection 4.2 for binary supports apply also when the supports for the bidders' valuations are three-point sets.

Proposition 6 *In the setting described in this section, consider the FPA with the Vickrey tie-breaking rule.*

(i) If $\min\{\lambda_H v_{2L} + (\lambda_L + \lambda_M) v_{1M}, (\lambda_M + \lambda_H) v_{2L} + \lambda_L v_{1L}\} \geq v_{1H}$*, then there exists a BNE in the FPA in which each type of bidder 1 bids the own valuation and each type of bidder 2 bids* v_{1H}*. In this case* $R^F = v_{1H}$ *is larger than* $R^S = \lambda_L v_{1L} + \lambda_M v_{1M} + \lambda_H v_{1H}$*.*

(ii) Suppose that $v_{1L} = v_{2L}$*,* $v_{1M} = v_{2M}$*, and for a given value of* v_{2H} *larger than* v_{2M}*, let* I *be a small interval centered in* v_{2H}*, that is* $I = (v_{2H} - \varepsilon, v_{2H} + \varepsilon)$ *for a small* $\varepsilon > 0$*. Then* R^F *is larger if* $v_{1H} = v_{2H}$ *than if* $v_{1H} \in I$ *and* $v_{1H} \neq v_{2H}$*. Furthermore,* $R^S > R^F$ *if* $v_{1H} \in I$ *and* $v_{1H} \neq v_{2H}$*.*

(iii) Suppose that $v_{2L} = v_{1L} + y\alpha$*,* $v_{2M} = v_{1M}$*,* $v_{2H} = v_{1H} - \alpha$ *for an arbitrary* $y > 0$ *and a small* $\alpha > 0$*. Then* $R^S > R^F$*.*

(iv) Suppose that $v_{2j} = v_{1j} + \alpha$ *for* $j = L, M, H$*, for a small* $\alpha > 0$*. Then* $R^S > R^F$*.*

For the case in which v_{2L} is sufficiently larger than v_{1H}, Proposition 6(i) describes a BNE for the FPA analogous to the BNE in Proposition 1(i). In this case $R^F > R^S$, as established by Proposition 2 for binary supports.

[32] Notice that $R^F > R^S$ requires v_{2H} sufficiently larger than v_{1H}, and jointly with $v_{1L} \leq v_{2L}$ and λ_1 sufficiently larger than λ_2, this implies that the distribution of v_2 first order stochastically dominates the distribution of v_1 strongly enough.

Proposition 6(ii) is analogous to Lemma 1 and to Proposition 4 [condition (9)], since it establishes that starting from a symmetric setting, a small increase in the valuation of type 1_H reduces R^F and thus makes R^F smaller than R^S. However, Proposition 6(ii) applies only for v_{1H} close to v_{2H}. The reason is that the effect of an increase of v_{1H} above v_{2H} is not immediate (whereas its effect is immediate for binary supports) since both types 1_M and 1_H become more aggressive; 2_L becomes less aggressive; the mixed strategy of type 2_H after the increase in v_{1H} is not comparable with his mixed strategy when $v_{1H} = v_{2H}$ in the sense of first order stochastic dominance. It is not straightforward to evaluate the net effect of these modified bidding strategies, thus Proposition 6(ii) restricts to the case of a small difference $v_{1H} - v_{2H}$, proving in particular that a small increase in v_{1H} above v_{2H} reduces R^F. Notice however that this implies a result analogous to Proposition 3: if we start from a symmetric setting such that $v_{1L} = v_{2L}$, $v_{1M} = v_{2M}$, $v_{1H} = v_{2H}$, then a suitable increase of all valuations reduces R^F.[33]

Proposition 6(iii) is analogous to Proposition 4 [condition (10)], as it shows that $R^S > R^F$ in a case such that v_2 is slightly less variable than v_1 (with $v_{1M} = v_{2M}$).

Finally, Proposition 6(iv) proves that $R^S > R^F$ for a small distribution shift, whereas a large shift makes the inequality in Proposition 6(i) satisfied, which implies $R^F > R^S$. Hence these results mirror exactly the results described in Subsection 4.2.3 on distribution shifts for binary supports.

6 Proof of Proposition 1

6.1 Proof of Proposition 1 for the case of $v_{1L} < v_{2L}$

For $i = 1, 2$ and $j = L, H$, let G_{ij} denote the c.d.f. for the mixed strategy of type j of bidder i, with $\underline{b}_{ij} = \inf\{b : G_{ij}(b) > 0\}$ and $\bar{b}_{ij} = \sup\{b : G_{ij}(b) < 1\}$. Recall that in a mixed-strategy BNE any bid made by type i_j must generate the same expected payoff, that is the equilibrium payoff of type i_j, which we denote by u^e_{ij}. We use $u_{ij}(b)$ and $p_{ij}(b)$ to denote the payoff of type i_j and his probability to win – respectively – as a function of his bid b, given the strategies of the two types of the other bidder.

This proof is organized in several steps, and throughout the proof ε denotes a number which is positive and close to zero. We start by recording a feature of any BNE.

Lemma 2 *If a profile of strategies has the property that there is a bid b' such that with a positive probability type 1_j and type 2_k tie bidding b' and* $\min\{v_{1j}, v_{2k}\} > b'$*, then the profile of strategies is not a BNE.*

Proof. By bidding b', at least one of these types loses the auction with positive probability; for instance type 1_j. Since $b' < v_{1j}$, type 1_j is better off bidding $b' + \varepsilon$ rather than b' as in this way his

[33] An instance in which this result is obtained is such that $v_{1L} = v_{2L} = 100$, $v_{1M} = v_{2M} = 200$, $v_{1H} = v_{2H} = 300$ and $\lambda_L = \lambda_M = \lambda_H = \frac{1}{3}$; then $R^F = \frac{1400}{9} = 155.\bar{5}$. However, if $v_{1L} = v_{2L} = 100.1$, $v_{1M} = v_{2M} = 200.1$, $v_{1H} = 310$ and $v_{2H} = 300.1$, then $R^F \simeq 155.475 < 155.\bar{5}$.

probability of winning increases discretely, whereas his payment in case of victory increases only slightly. ■

6.1.1 Step 1: When $v_{1L} < v_{2L}$, any BNE is such that (i) $\bar{b}_{1L} \leq \underline{b}_{1H}$, $\bar{b}_{2L} \leq \underline{b}_{2H}$; (ii) either $\underline{b}_{1L} = \underline{b}_{2L} = v_{1L} = \bar{b}_{1L}$ or $\underline{b}_{1L} < \underline{b}_{2L}$; (iii) $u^e_{1L} = 0$, $u^e_{2L} > 0$, $v_{1L} \leq \underline{b}_{2L}$; (iv) $\bar{b}_{1H} = \bar{b}_{2H}$

(i) The monotonicity properties $\bar{b}_{1L} \leq \underline{b}_{1H}$ and $\bar{b}_{2L} \leq \underline{b}_{2H}$ follow from Proposition 1 in Maskin and Riley (2000b).

(ii) In order to prove that $\underline{b}_{1L} \leq \underline{b}_{2L}$, suppose in view of a contradiction that $\underline{b}_{2L} < \underline{b}_{1L}$. Since 2_L bids in the interval $[\underline{b}_{2L}, \underline{b}_{1L})$ with positive probability, it follows that $u^e_{2L} = 0$. However, since $\underline{b}_{1L} \leq v_{1L} < v_{2L}$ we find that $p_{2L}(b) > 0$ and $u_{2L}(b) > 0$ if 2_L bids $b = \underline{b}_{1L} + \varepsilon$: contradiction.

We now show that if $\underline{b}_{1L} = \underline{b}_{2L} \equiv \underline{b}$, then $\underline{b} = v_{1L}$, and as a consequence we obtain $\bar{b}_{1L} = v_{1L}$. Suppose $\underline{b} < v_{1L}$. We distinguish several cases depending on whether G_{1L} and/or G_{2L} puts an atom on $\underline{b}$; in each case we obtain a contradiction.

- $G_{1L}(\underline{b}) = 0$ [$G_{2L}(\underline{b}) = 0$ or $G_{2L}(\underline{b}) > 0$ does not matter]. In this case $u^e_{2L} = 0$ as $p_{2L}(b)$ is about zero for b close to $\underline{b}$ (as G_{1L} is right continuous). However, since $\underline{b} < v_{1L} < v_{2L}$ we find that $p_{2L}(b) > 0$ and $u_{2L}(b) > 0$ if 2_L bids $b = \underline{b} + \varepsilon$.
- $G_{1L}(\underline{b}) > 0$ and $G_{2L}(\underline{b}) > 0$. This case is ruled out by Lemma 2.
- $G_{1L}(\underline{b}) > 0$ and $G_{2L}(\underline{b}) = 0$. In this case $u^e_{1L} = 0$ as $p_{1L}(\underline{b}) = 0$. However, since $\underline{b} < v_{1L}$ we find that $p_{1L}(b) > 0$ and $u_{1L}(b) > 0$ if 1_L bids $b = \underline{b} + \varepsilon < v_{1L}$.

(iii) We notice that $u^e_{1L} = 0$ both if $\underline{b}_{1L} = \underline{b}_{2L} = \bar{b}_{1L} = v_{1L}$ and if $\underline{b}_{1L} < \underline{b}_{2L}$. Hence $v_{1L} \leq \underline{b}_{2L}$, since if $\underline{b}_{2L} < v_{1L}$ then any bid in $(\underline{b}_{2L}, v_{1L})$ yields a positive payoff to 1_L. Finally, $p_{2L}(b) \geq \lambda_1$ for any $b \geq v_{1L} + \varepsilon$, thus $u^e_{2L} \geq \lambda_1(v_{2L} - v_{1L} - \varepsilon) > 0$ for each small $\varepsilon > 0$.

(iv) If $\bar{b}_{1H} > \bar{b}_{2H}$, then it is profitable for 1_H to move some probability from $(\bar{b}_{1H} - \varepsilon, \bar{b}_{1H}]$ to $(\bar{b}_{2H}, \bar{b}_{2H} + \varepsilon)$, since the probability of winning remains 1 but his payment in case of victory is smaller. If $\bar{b}_{1H} < \bar{b}_{2H}$, a symmetric argument applies to 2_H.

6.1.2 Step 2: When $v_{1L} < v_{2L}$, there exists a BNE such that $\bar{b}_{1H} \leq \underline{b}_{2L}$ if and only if (3) is satisfied; any such BNE is outcome-equivalent to the BNE in Proposition 1(i)

We start by proving that $\underline{b}_{1L} < \underline{b}_{2L}$. Suppose in view of a contradiction that $\underline{b}_{1L} = \underline{b}_{2L}$. Then Step 1(i-ii) imply $\underline{b}_{1L} = \bar{b}_{1L} = \underline{b}_{1H} = \bar{b}_{1H} = \underline{b}_{2L} = v_{1L}$. It is impossible that $G_{2L}(v_{1L}) > 0$, because in such a case 1_H and 2_L would tie with positive probability at $b = v_{1L}$, and then Lemma 2 would apply. As a consequence, $p_{1H}(v_{1L}) = 0$ and $u^e_{1H} = 0$. However, if 1_H plays $b = v_{1L} + \varepsilon$ then $p_{1H}(b) > 0$ and $u_{1H}(b) > 0$ since $v_{1L} < v_{1H}$: contradiction.

From the inequality $\bar{b}_{1H} \leq \underline{b}_{2L}$ it follows that 2_L wins with probability one;[34] thus $u^e_{1H} = 0$. Moreover, (i) $\bar{b}_{1H} = \bar{b}_{2H}$ by Step 1(iv) and thus $\bar{b}_{1H} = \underline{b}_{2L} = \bar{b}_{2L} = \underline{b}_{2H} = \bar{b}_{2H}$; (ii) $v_{1H} \leq \underline{b}_{2L}$ otherwise any bid in $(\underline{b}_{2L}, v_{1H})$ yields a positive payoff to 1_H. Hence, $u^e_{2L} = v_{2L} - \underline{b}_{2L}$ and $u^e_{2H} = v_{2H} - \underline{b}_{2L}$.

We need to examine the incentives of bidder 2 to bid below $\underline{b}_{2L}$, and in particular we notice that bidding $b = \bar{b}_{1L} + \varepsilon$ yields bidder 2 a probability of winning not smaller than λ_1. Thus the inequalities

$$\lambda_1(v_{2L} - \bar{b}_{1L} - \varepsilon) \leq v_{2L} - \underline{b}_{2L} \qquad \text{and} \qquad \lambda_1(v_{2H} - \bar{b}_{1L} - \varepsilon) \leq v_{2H} - \underline{b}_{2L}$$

need to hold for any $\varepsilon > 0$, and since $v_{2H} > v_{2L}$ it is simple to see that the first inequality is more restrictive than the second one. Given $\bar{b}_{1L} \leq v_{1L}$ and $\underline{b}_{2L} \geq v_{1H}$, the first inequality is most likely to be satisfied when $\bar{b}_{1L} = v_{1L}$ and $\underline{b}_{2L} = v_{1H}$, and then it reduces to (3). This inequality is therefore a necessary condition for the existence of a BNE such that $\bar{b}_{1H} \leq \underline{b}_{2L}$.

Bids above v_{1H} are obviously suboptimal for bidder 2 because $u_{2L}(b) = v_{2L} - b < v_{2L} - v_{1H}$ if $b > v_{1H}$. On the other hand, for bids smaller than v_{1H} the strategies of 1_L and 1_H need to be such that no $b < v_{1H}$ is a profitable deviation for type 2_L.[35] For instance, we verify that this condition is satisfied if G_{1H} is the uniform distribution over $[\alpha v_{1H}, v_{1H}]$, with $\alpha < 1$ and close to 1; recall that 1_L bids v_{1L} with certainty. Then $p_{2L}(b) = 0$, $u_{2L}(b) = 0$ for $b < v_{1L}$, whereas $p_{2L}(v_{1L}) = \lambda_1$ (recall the Vickrey tie-breaking rule and $v_{2L} > v_{1L}$), $u_{2L}(v_{1L}) = \lambda_1(v_{2L} - v_{1L})$, but we know from (3) that this payoff is smaller than $v_{2L} - v_{1H}$, the payoff of 2_L if he bids v_{1H}. For $b \in (v_{1L}, \alpha v_{1H})$ we find that $u_{2L}(b) = \lambda_1(v_{2L} - b)$ is decreasing. Finally, for $b \in [\alpha v_{1H}, v_{1H}]$, $u_{2L}(b) = (v_{2L} - b)[\lambda_1 + (1-\lambda_1)\frac{b-\alpha v_{1H}}{v_{1H}-\alpha v_{1H}}]$ and is increasing for $\alpha > 1 - \frac{(1-\lambda)(v_{2L}-v_{1H})}{v_{1H}}$, which implies that $b = v_{1H}$ is a best reply for 2_L.

6.1.3 Step 3: When $v_{1L} < v_{2L}$, there exists no BNE such that $\underline{b}_{2L} < \bar{b}_{1H} \leq \bar{b}_{2L}$

If $\underline{b}_{2L} < \bar{b}_{1H} \leq \bar{b}_{2L}$, then $\underline{b}_{2L} < \bar{b}_{1H} = \bar{b}_{2L} = \underline{b}_{2H} = \bar{b}_{2H} \equiv b^*$ by Step 1(iv). This implies $b^* \leq v_{1H}$, and thus $\underline{b}_{2L} < b^*$ implies $u^e_{1H} > 0$, and in turn $b^* < v_{1H}$. Since 2_H bids b^* with certainty, it is profitable for 1_H to bid $b^* + \varepsilon$ rather than $b^* - \varepsilon$, as in this way his probability of victory increases by at least $1 - \lambda_2 > 0$ and his payment in case of victory increases only slightly.

6.1.4 Step 4: When $v_{1L} < v_{2L}$, there exists a BNE such that $\underline{b}_{2L} < \bar{b}_{2L} < \bar{b}_{1H}$ if and only if (4) is satisfied; any such BNE is outcome-equivalent to the BNE in Proposition 1(ii)

The inequality $\underline{b}_{2L} < \bar{b}_{1H}$ implies $u^e_{1H} > 0$ because $\bar{b}_{1H} \leq v_{1H}$ and $p_{1H}(b) > 0$ for $b \in (\underline{b}_{2L}, \bar{b}_{1H})$. Next lemma provides a list of features of any BNE such that $\underline{b}_{2L} < \bar{b}_{1H}$.

[34] In particular, if $\bar{b}_{1H} = \underline{b}_{2L}$ and 1_H and 2_L tie with positive probability at $\underline{b}_{2L}$, then 2_L needs to win the tie-break with probability 1, otherwise it is profitable for him to bid $\underline{b}_{2L} + \varepsilon$ rather than $\underline{b}_{2L}$ ($\underline{b}_{2L} < v_{2L}$ since $u^e_{2L} > 0$).

[35] If this property is satisfied, then no deviation is profitable for 2_H since $(v_{2L} - b)p_{2L}(b) \leq v_{2L} - v_{1H}$ implies $(v_{2H} - b)p_{2H}(b) \leq v_{2H} - v_{1H}$, as $p_{2L}(b) = p_{2H}(b)$ for any b

Lemma 3 *In any BNE such that* $\underline{b}_{2L} < \bar{b}_{1H}$ *the following equalities hold:* $\bar{b}_{1L} = \underline{b}_{1H} = \underline{b}_{2L} = v_{1L}$, $\bar{b}_{2L} = \underline{b}_{2H}$*; moreover,* $G_{2L}(\underline{b}_{2L}) > 0$.

Proof. The proof is split in two claims.

Claim 1 $\bar{b}_{1L} = \underline{b}_{1H}$.

In view of a contradiction, assume that $\bar{b}_{1L} < \underline{b}_{1H}$. If $G_{1H}(\underline{b}_{1H}) > 0$ and $G_{2L}(\underline{b}_{1H}) > 0$,[36] then Lemma 2 applies since $u^e_{1H} > 0$ and $u^e_{2L} > 0$ imply $v_{1H} > \underline{b}_{1H}$ and $v_{2L} > \underline{b}_{1H}$. If $G_{1H}(\underline{b}_{1H}) > 0$ and 2 puts no atom at $\underline{b}_{1H}$, then 2 bids with zero probability in $(\bar{b}_{1L} + \varepsilon, \underline{b}_{1H}]$ and 1_H can increase his payoff by moving the atom from $\underline{b}_{1H}$ to any point in $(\bar{b}_{1L} + \varepsilon, \underline{b}_{1H}]$. If $G_{1H}(\underline{b}_{1H}) = 0$, then 2 bids with zero probability in $(\bar{b}_{1L} + \varepsilon, \underline{b}_{1H}]$ (in particular, 2 puts no atom in $\underline{b}_{1H}$) and then 1_H can increase his payoff by moving some probability from $[\underline{b}_{1H}, \underline{b}_{1H} + \varepsilon)$ to $(\bar{b}_{1L} + \varepsilon, \bar{b}_{1L} + 2\varepsilon)$.

Claim 2 $\underline{b}_{1H} = \underline{b}_{2L} = v_{1L}$, $G_{2L}(v_{1L}) > 0$ and $\bar{b}_{2L} = \underline{b}_{2H}$.

If $\underline{b}_{1H} < \underline{b}_{2L}$, then 1_H bids in $[\underline{b}_{1H}, \underline{b}_{2L})$ with positive probability and thus $u^e_{1H} = 0$: contradiction. Thus $\underline{b}_{2L} \leq \underline{b}_{1H}$ and since $\bar{b}_{1L} \leq v_{1L}$, $v_{1L} \leq \underline{b}_{2L}$ [by Step 1(iii)] and $\bar{b}_{1L} = \underline{b}_{1H}$ (by Claim 1), we infer that $\bar{b}_{1L} = \underline{b}_{2L} = \underline{b}_{1H} = v_{1L}$. Moreover, given $\underline{b}_{1H} = \underline{b}_{2L}$, if $G_{2L}(\underline{b}_{2L}) = 0$ then $u^e_{1H} = 0$; thus $G_{2L}(\underline{b}_{2L}) > 0$. The equality $\bar{b}_{2L} = \underline{b}_{2H}$ is proved along the same lines followed in Claim 1 to prove $\bar{b}_{1L} = \underline{b}_{1H}$. ■

Lemma 4 *In any BNE such that* $\underline{b}_{2L} < \bar{b}_{2L} < \bar{b}_{1H}$*, the mixed strategies of* 1_H*,*2_L*,*2_H *are given by (5)-(6), and they constitute a BNE if and only if (4) is satisfied.*

Proof. In the following of this proof we use $\hat{b}$ and $\bar{b}$, respectively, instead of $\bar{b}_{2L}$ and of $\bar{b}_{2H} = \bar{b}_{1H}$. Given that $v_{1L} < \hat{b}$, types $1_H, 2_L, 2_H$ are all employing mixed strategies and we can argue like in the proof of Claim 1 in Lemma 2 to show that G_{1H}, G_{2L}, G_{2H} are strictly increasing and continuous in the intervals $[v_{1L}, \bar{b}]$, $[v_{1L}, \hat{b}]$, $[\hat{b}, \bar{b}]$, respectively. This implies that the following conditions must be satisfied.

Indifference condition of type 1_H:

$$(v_{1H} - b)[\lambda_2 G_{2L}(b) + (1 - \lambda_2) G_{2H}(b)] = v_{1H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in (v_{1L}, \bar{b}] \tag{15}$$

Indifference condition of type 2_L:

$$(v_{2L} - b)[\lambda_1 + (1 - \lambda_1) G_{1H}(b)] = \lambda_1 (v_{2L} - v_{1L}) \qquad \text{for} \quad \text{any} \quad b \in [v_{1L}, \hat{b}] \tag{16}$$

Indifference condition of type 2_H:

$$(v_{2H} - b)[\lambda_1 + (1 - \lambda_1) G_{1H}(b)] = v_{2H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in [\hat{b}, \bar{b}] \tag{17}$$

From (16) and (17) we obtain G_{1H} in (5). For $b \in (v_{1L}, \hat{b}]$, (15) reduces to $(v_{1H} - b)\lambda_2 G_{2L}(b) = v_{1H} - \bar{b}$ and thus G_{2L} satisfies (6). For $b \in [\hat{b}, \bar{b}]$, (15) reduces to $(v_{1H} - b)[\lambda_2 + (1 - \lambda_2) G_{2H}(b)] = v_{1H} - \bar{b}$ and then G_{2H} satisfies (6).

[36] If we consider type 2_H instead of 2_L, the same the argument applies.

Since $G_{2L}(\hat{b}) = 1$, we deduce that $\bar{b} = \lambda_2 \hat{b} + (1-\lambda_2)v_{1H}$, and since G_{1H} needs to be continuous at $b = \hat{b}$ we infer that $\hat{b}$ solves (2); here we use $Z(b)$ to denote the left hand side of (2). The strategies in Proposition 1(ii) require that $\hat{b}$ satisfies $v_{1L} < \hat{b} < \min\{v_{2L}, v_{1H}\}$, and since $Z(v_{2L}) = -\lambda_1 (v_{2L} - v_{1L})(v_{2H} - v_{2L}) < 0$ we infer that $\hat{b}$ is the smaller solution of (2); moreover, $Z(v_{1L}) = (1-\lambda_2)(v_{2L} - v_{1L})\left(\frac{(\lambda_1-\lambda_2)v_{1L}+(1-\lambda_1)v_{2H}}{1-\lambda_2} - v_{1H}\right)$ and thus $\frac{(\lambda_1-\lambda_2)v_{1L}+(1-\lambda_1)v_{2H}}{1-\lambda_2} > v_{1H}$ needs to hold. The inequality $\hat{b} < v_{1H}$ is obviously satisfied if $v_{2L} \leq v_{1H}$, while if $v_{1H} < v_{2L}$ then it is equivalent to $Z(v_{1H}) < 0$. Since $Z(v_{1H}) = -[v_{1H} - \lambda_1 v_{1L} - (1-\lambda_1)v_{2L}](v_{2H} - v_{1H})$ and $v_{1H} < v_{2L} < v_{2H}$, we deduce that the converse of (3) needs to hold. Thus (4) is a necessary condition for the existence of a BNE such that $\underline{b}_{2L} < \bar{b}_{2L} < \bar{b}_{1H}$.

Now we verify that for each type of each bidder the strategy specified by Proposition 1(ii) is a best reply given the strategies of the two types of the other bidder. Notice that $p_{1H}(\bar{b}) = p_{2H}(\bar{b}) = 1$, thus we do not need to consider bids above $\bar{b}$. The same remark applies to the BNE described by Proposition 1(iii).

Type 1_L. The strategies of types 2_L and 2_H are such that each type of bidder 2 bids at least v_{1L} with probability one. Therefore the payoff of 1_L is zero if he bids v_{1L} as specified by Proposition 1, and it is impossible for him to obtain a positive payoff.

Type 1_H. We know from (15) that the payoff of 1_H is $v_{1H} - \bar{b} > 0$ for any $b \in (v_{1L}, \bar{b}]$. If $b < v_{1L}$, then $p_{1H}(b) = 0$ and $u_{1H}(b) = 0$. If $b = v_{1L}$, then 1_H loses against 2_H and loses also against 2_L unless 2_L bids v_{1L}, in which case 1_H ties with 2_L – an event with probability $G_{2L}(v_{1L})$. Consider the most favorable case for 1_H, which means that he wins the tie-break against 2_L with probability one (this occurs if $v_{2L} < v_{1H}$): his expected payoff from bidding v_{1L} is then $(v_{1H} - v_{1L})\lambda_2 G_{2L}(v_{1L})$, which turns out to be equal to $v_{1H} - \bar{b}$.

Type 2_L. We know from (16) that the payoff of 2_L is $\lambda_1(v_{2L} - v_{1L}) > 0$ for any $b \in [v_{1L}, \hat{b}]$. For bids smaller than v_{1L}, the payoff of 2_L is zero as $p_{2L}(b) = 0$ if $b < v_{1L}$. If $b \in [\hat{b}, \bar{b}]$, then $u_{2L}(b) = (v_{2L} - b)[\lambda_1 + (1-\lambda_1)G_{1H}(b)] = (v_{2L} - b)\frac{v_{2H}-\bar{b}}{v_{2H}-b}$ which is decreasing in b, and therefore $u_{2L}(\hat{b}) > u_{2L}(b)$ for any $b \in (\hat{b}, \bar{b}]$.

Type 2_H. We know from (17) that the payoff of 2_H is $v_{2H} - \bar{b} > 0$ for any $b \in [\hat{b}, \bar{b}]$. For bids smaller than v_{1L}, the payoff of 2_H is zero as $p_{2H}(b) = 0$ if $b < v_{1L}$. If $b \in [v_{1L}, \hat{b}]$, then $p_{2H}(b) = \lambda_1 + (1-\lambda_1)G_{1H}(b) = \lambda_1 \frac{v_{2L}-v_{1L}}{v_{2L}-b}$ and $u_{2H}(b) = (v_{2H} - b)\lambda_1 \frac{v_{2L}-v_{1L}}{v_{2L}-b}$, which is increasing in b and therefore $u_{2H}(b) < u_{2H}(\hat{b})$ for any $b \in [v_{1L}, \hat{b})$. ■

6.1.5 Step 5: When $v_{1L} < v_{2L}$, there exists a BNE such that $\underline{b}_{2L} = \bar{b}_{2L} < \bar{b}_{1H}$ if and only if (7) is satisfied; any such BNE is outcome-equivalent to the BNE in Proposition 1(iii)

In this case Lemma 3 (in the proof of Step 4) applies, thus we infer that $\bar{b}_{1L} = \underline{b}_{1H} = \underline{b}_{2L} = \bar{b}_{2L} = \underline{b}_{2H} = v_{1L}$; this means that 2_L plays a pure strategy and bids v_{1L}. Conversely, types 1_H and 2_H employ mixed strategies and thus the following indifference conditions need to hold, in which we still use $\bar{b}$ instead of $\bar{b}_{2H} = \bar{b}_{1H}$. For type 1_H:

$$(v_{1H} - b)[\lambda_2 + (1 - \lambda_2)G_{2H}(b)] = v_{1H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in (v_{1L}, \bar{b}] \tag{18}$$

For type 2_H:

$$(v_{2H} - b)[\lambda_1 + (1 - \lambda_1)G_{1H}(b)] = v_{2H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in (v_{1L}, \bar{b}] \tag{19}$$

Notice that $G_{1H}(v_{1L}) = 0$ since if $G_{1H}(v_{1L}) > 0$, then 1_H ties with 2_L with positive probability by bidding v_{1L}, and thus Lemma 2 applies. From $G_{1H}(v_{1L}) = 0$ and (19) we obtain $\bar{b} = \lambda_1 v_{1L} + (1 - \lambda_1)v_{2H}$, and then (18)-(19) yield G_{1H}, G_{2H} in (8). The inequality (7) needs to hold since it is equivalent to $G_{2H}(v_{1L}) \geq 0$.

Now we verify that for each type of each bidder the strategy specified by Proposition 1(iii) is a best reply given the strategies of the two types of the other bidder.

Type 1_L. The same argument given in the proof of Lemma 4 in Step 4 applies.

Type 1_H. We know from (18) that the payoff of 1_H is $v_{1H} - \bar{b} > 0$ for any $b \in (v_{1L}, \bar{b}]$,[37] and $b < v_{1L}$ implies $p_{1H}(b) = 0$, $u_{1H}(b) = 0$. If $b = v_{1L}$, then 1_H ties with type 2_L and loses against 2_H, unless also 2_H bids v_{1L} – an event with probability $G_{2H}(v_{1L})$. Consider the most favorable case for 1_H, which means that he wins the tie-break against each type of bidder 2 with probability one (this occurs if $v_{2H} < v_{1H}$): his expected payoff from bidding v_{1L} is then $(v_{1H} - v_{1L})[\lambda_2 + (1 - \lambda_2)G_{2H}(v_{1L})]$ which turns out to be equal to $v_{1H} - \bar{b}$.

Type 2_L. The payoff of 2_L is $\lambda_1(v_{2L} - v_{1L})$. For bids smaller than v_{1L} we can argue exactly like in the proof of Lemma 4 in Step 4. If $b \in [v_{1L}, \bar{b}]$, then $p_{2L}(b) = \lambda_1 \frac{v_{2H} - v_{1L}}{v_{2H} - b}$ and thus $u_{2L}(b) = (v_{2L} - b)\lambda_1 \frac{v_{2H} - v_{1L}}{v_{2H} - b}$ is decreasing in b.

Type 2_H. The payoff of 2_H is $v_{2H} - \bar{b} > 0$ for any $b \in [v_{1L}, \bar{b}]$. For bids smaller than v_{1L} we can argue exactly like in the proof of Lemma 4 in Step 4.

6.2 Proof of Proposition 1 for the case of $v_{1L} = v_{2L}$

6.2.1 Step 1: When $v_{1L} = v_{2L} = v_L$, any BNE is such that $\underline{b}_{1L} = \underline{b}_{2L} = \bar{b}_{1L} = \bar{b}_{2L} = v_L$

We start by proving that $\underline{b}_{1L} = \underline{b}_{2L}$. In view of a contradiction, suppose that $\underline{b}_{2L} < \underline{b}_{1L}$. Since 2_L bids in $[\underline{b}_{2L}, \underline{b}_{1L})$ with positive probability, it follows that $u^e_{2L} = 0$. Then $v_L \leq \underline{b}_{1L}$, since $\underline{b}_{1L} < v_L$ implies that $p_{2L}(b) > 0$ and $u_{2L}(b) > 0$ for any $b \in (\underline{b}_{1L}, v_L)$. Moreover, $v_L \leq \underline{b}_{1L}$ implies $u^e_{1L} = 0$, but $p_{1L}(b) > 0$ and $u_{1L}(b) > 0$ for any $b \in (\underline{b}_{2L}, \underline{b}_{1L})$: contradiction. Therefore the inequality $\underline{b}_{2L} < \underline{b}_{1L}$ cannot hold in equilibrium, and a similar argument applies to rule out $\underline{b}_{1L} < \underline{b}_{2L}$.

Given that $\underline{b}_{1L} = \underline{b}_{2L} \equiv \underline{b}$, we prove that $\underline{b} = v_L$. In view of a contradiction, suppose that $\underline{b} < v_L$. In case that $G_{1L}(\underline{b}) > 0$ and $G_{2L}(\underline{b}) > 0$, Lemma 2 applies; thus $G_{1L}(\underline{b}) = 0$ and/or $G_{2L}(\underline{b}) = 0$. If $G_{1L}(\underline{b}) = 0$, we find that $u^e_{2L} = 0$ since $p_{2L}(b)$ is about 0 for b close to $\underline{b}$, but in fact 2_L can make a positive payoff by bidding in $(\underline{b}, v_L)$: contradiction. The same argument applies if $G_{2L}(\underline{b}) = 0$. Thus $\underline{b} = v_L$, which implies $\bar{b}_{1L} = \bar{b}_{2L} = v_L$: hence both 1_L and 2_L bid v_L with probability one.

[37] Notice that $v_{1H} - \bar{b} > 0$ given (7).

6.2.2 Step 2: When $v_{1L} = v_{2L} = v_L$, in the unique BNE 1_H,2_H play the mixed strategies described by Proposition 1(iii) if (7) holds; if (7) is violated, then 1_H,2_H play the mixed strategies described by (5) and (6) with $\hat{b} = v_L$

As in the proof of Proposition 1(ii) (Lemma 3 in Step 4) we can prove that $\bar{b}_{1L} = \underline{b}_{1H} (= v_L)$ and $\bar{b}_{2L} = \underline{b}_{2H} (= v_L)$. Using again $\bar{b}$ instead of $\bar{b}_{1H}, \bar{b}_{2H}$ we infer that G_{1H}, G_{2H} need to satisfy

$$(v_{1H} - b)[\lambda_2 + (1 - \lambda_2)G_{2H}(b)] = v_{1H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in [v_L, \bar{b}] \tag{20}$$

and

$$(v_{2H} - b)[\lambda_1 + (1 - \lambda_1)G_{1H}(b)] = v_{2H} - \bar{b} \qquad \text{for} \quad \text{any} \quad b \in [v_L, \bar{b}] \tag{21}$$

From (20)-(21) we obtain $G_{1H}(v_L) = \frac{1}{1-\lambda_1}(\frac{v_{2H}-\bar{b}}{v_{2H}-v_L} - \lambda_1)$ and $G_{2H}(v_L) = \frac{1}{1-\lambda_2}(\frac{v_{1H}-\bar{b}}{v_{1H}-v_L} - \lambda_2)$. Lemma 2 implies that $G_{1H}(v_L) > 0$ and $G_{2H}(v_L) > 0$ cannot hold. Thus we consider the other cases.

If $G_{1H}(v_L) > 0 = G_{2H}(v_L)$ we obtain $\bar{b} = \lambda_2 v_L + (1 - \lambda_2)v_{1H}$ and $G_{1H}(v_L) > 0$ is equivalent to the converse of (7); from (20)-(21) we obtain G_{1H}, G_{2H} as in footnote 14.[38] Now we prove that no profitable deviation exists for any type. The payoff of 1_L (2_L) is zero and he needs to bid above v_L in order to win. For 1_H, we know from (20) that his payoff is $v_{1H} - \bar{b}$ for any $b \in [v_L, \bar{b}]$ and $b < v_L$ yields $u_{1H}(b) = 0$. A similar argument applies to 2_H.

In case that $G_{2H}(v_L) \geq 0 = G_{1H}(v_L)$ we obtain $\bar{b} = \lambda_1 v_L + (1 - \lambda_1)v_{2H}$, and $G_{2H}(v_{1L}) \geq 0$ is equivalent to (7); from (20)-(21) we obtain G_{1H}, G_{2H} as in (8). The proof that no profitable deviation exists for any type is exactly as when (7) is violated.

6.3 Derivation of R^F given the BNE described by Proposition 1

6.3.1 The BNE of Proposition 1(ii) when $v_{1L} < v_{2L}$

We evaluate R^F as the difference between the social surplus S^F generated by the FPA minus the bidders' rents U^F: $R^F = S^F - U^F$. Thus we need to derive S^F and U^F:

$$\begin{aligned} S^F &= \lambda_1\lambda_2 v_{2L} + \lambda_1(1-\lambda_2)v_{2H} + (1-\lambda_1)\lambda_2[v_{2L} + (v_{1H} - v_{2L})\Pr\{1_H \text{ def } 2_L\}] \\ &\quad +(1-\lambda_1)(1-\lambda_2)[v_{2H} + (v_{1H} - v_{2H})\Pr\{1_H \text{ def } 2_H\}] \end{aligned}$$

and

$$U^F = (1-\lambda_1)(v_{1H} - \lambda_2\hat{b} - (1-\lambda_2)v_{1H}) + (1-\lambda_2)(v_{2H} - \lambda_2\hat{b} - (1-\lambda_2)v_{1H}) + \lambda_2\lambda_1(v_{2L} - v_{1L})$$

[38] Step 1 and the proof of Step 2 up to this point apply for any tie-breaking rule. However, no BNE exists under the standard tie-breaking rule if (7) is violated since (i) $G_{1H}(v_L) > 0$ and 1_H and 2_L tie with positive probability at the bid v_L; (ii) it is profitable for 1_H to bid $v_L + \varepsilon$ rather than v_L, which breaks the BNE [a similar argument applies if (7) holds with strict inequality]. On the other hand, with the Vickrey tie-breaking rule we have $c_{1H} = v_{1H} - v_L > 0$ and $c_{2L} = 0$; thus 1_H wins (paying v_L as aggregate price) in case of tie with 2_L.

in which $\Pr\{1_H \text{ def } 2_j\}$, for $j = L, H$, is the probability that 1_H wins when he faces type 2_j. Therefore

$$\begin{aligned} R^F &= \lambda_2(2-\lambda_1-\lambda_2)\hat{b} + (1+\lambda_2^2+\lambda_1\lambda_2-3\lambda_2)v_{1H} + \lambda_2(1-\lambda_1)v_{2L} + \lambda_2\lambda_1 v_{1L} \\ &\quad +(1-\lambda_1)\lambda_2(v_{1H}-v_{2L})\Pr\{1_H \text{ def } 2_L\} + (1-\lambda_1)(1-\lambda_2)(v_{1H}-v_{2H})\Pr\{1_H \text{ def } 2_H\} \end{aligned}$$

Derivation of $\Pr\{1_H$ **def** $2_L\}$ For the case that $v_{1H} \neq v_{2L}$ we need to evaluate

$$\Pr\{1_H \text{ def } 2_L\} = \int_{v_{1L}}^{\hat{b}} G'_{1H}(b)G_{2L}(b)db + 1 - G_{1H}(\hat{b})$$

and using $\bar{b} = \lambda_2\hat{b} + (1-\lambda_2)v_{1H}$ in G_{2L} we find $G_{2L}(b) = \frac{v_{1H}-\hat{b}}{v_{1H}-b}$:

$$\begin{aligned} \Pr\{1_H \text{ def } 2_L\} &= \int_{v_{1L}}^{\hat{b}} \frac{\lambda_1}{1-\lambda_1}\frac{v_{2L}-v_{1L}}{(v_{2L}-b)^2}\frac{v_{1H}-\hat{b}}{v_{1H}-b}db + 1 - \frac{\lambda_1(\hat{b}-v_{1L})}{(1-\lambda_1)(v_{2L}-\hat{b})} \\ &= \frac{\lambda_1(v_{2L}-v_{1L})(v_{1H}-\hat{b})}{1-\lambda_1}\int_{v_{1L}}^{\hat{b}} \frac{1}{(v_{2L}-b)^2(v_{1H}-b)}db + 1 - \frac{\lambda_1(\hat{b}-v_{1L})}{(1-\lambda_1)(v_{2L}-\hat{b})} \end{aligned}$$

We exploit

$$\int \frac{1}{(v_{2L}-b)^2(v_{1H}-b)}db = \frac{1}{(v_{1H}-v_{2L})^2}\ln\left|\frac{v_{2L}-b}{v_{1H}-b}\right| + \frac{1}{(v_{1H}-v_{2L})(v_{2L}-b)}$$

to obtain

$$\int_{v_{1L}}^{\hat{b}} \frac{1}{(v_{2L}-b)^2(v_{1H}-b)}db = \frac{1}{(v_{1H}-v_{2L})^2}\ln\frac{(v_{2L}-\hat{b})(v_{1H}-v_{1L})}{(v_{1H}-\hat{b})(v_{2L}-v_{1L})} + \frac{\hat{b}-v_{1L}}{(v_{1H}-v_{2L})(v_{2L}-\hat{b})(v_{2L}-v_{1L})}$$

thus

$$\Pr\{1_H \text{ def } 2_L\} = \frac{\lambda_1(v_{1H}-\hat{b})(v_{2L}-v_{1L})}{(1-\lambda_1)(v_{1H}-v_{2L})^2}\ln\frac{(v_{2L}-\hat{b})(v_{1H}-v_{1L})}{(v_{1H}-\hat{b})(v_{2L}-v_{1L})} + \frac{(1-\lambda_1)(v_{1H}-v_{2L}) + \lambda_1(\hat{b}-v_{1L})}{(1-\lambda_1)(v_{1H}-v_{2L})}$$

and

$$\begin{aligned} (1-\lambda_1)\lambda_2(v_{1H}-v_{2L})\Pr\{1_H \text{ def } 2_L\} &= \frac{\lambda_1\lambda_2(v_{1H}-\hat{b})(v_{2L}-v_{1L})}{v_{1H}-v_{2L}}\ln\frac{(v_{2L}-\hat{b})(v_{1H}-v_{1L})}{(v_{1H}-\hat{b})(v_{2L}-v_{1L})} \\ &\quad +\lambda_2(1-\lambda_1)(v_{1H}-v_{2L}) + \lambda_1\lambda_2(\hat{b}-v_{1L}) \end{aligned}$$

Derivation of $\Pr\{1_H$ **def** $2_H\}$ For the case that $v_{1H} \neq v_{2H}$ we need to evaluate

$$\Pr\{1_H \text{ def } 2_H\} = \int_{\hat{b}}^{\bar{b}} G'_{1H}(b)G_{2H}(b)db$$

and using $\bar{b} = \lambda_2\hat{b} + (1-\lambda_2)v_{1H}$ in G_{2H} we find $G_{2H}(b) = \frac{\lambda_2(b-\hat{b})}{(1-\lambda_2)(v_{1H}-b)}$:

$$\begin{aligned} \Pr\{1_H \text{ def } 2_H\} &= \int_{\hat{b}}^{\bar{b}} \frac{v_{2H}-\bar{b}}{(1-\lambda_1)(v_{2H}-b)^2}\frac{\lambda_2(b-\hat{b})}{(1-\lambda_2)(v_{1H}-b)}db \\ &= \frac{\lambda_2(v_{2H}-\bar{b})}{(1-\lambda_1)(1-\lambda_2)}\int_{\hat{b}}^{\bar{b}} \frac{b-\hat{b}}{(v_{1H}-b)(v_{2H}-b)^2}db \end{aligned}$$

We exploit

$$\int \frac{b-\hat{b}}{(v_{1H}-b)(v_{2H}-b)^2}db = \frac{v_{1H}-\hat{b}}{(v_{1H}-v_{2H})^2}\ln\left|\frac{v_{2H}-b}{v_{1H}-b}\right| - \frac{v_{2H}-\hat{b}}{(v_{2H}-v_{1H})(v_{2H}-b)}$$

to obtain

$$\begin{aligned}\int_{\hat{b}}^{\bar{b}} \frac{b-\hat{b}}{(v_{1H}-b)(v_{2H}-b)^2}db &= \frac{v_{1H}-\hat{b}}{(v_{1H}-v_{2H})^2}\ln\frac{v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H}}{\lambda_2(v_{2H}-\hat{b})} \\ &-\frac{(1-\lambda_2)(v_{1H}-\hat{b})}{(v_{2H}-v_{1H})\left(v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H}\right)}\end{aligned}$$

thus

$$\begin{aligned}\Pr\{1_H \text{ def } 2_H\} &= \frac{\lambda_2(v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H})}{(1-\lambda_1)(1-\lambda_2)}\frac{v_{1H}-\hat{b}}{(v_{1H}-v_{2H})^2}\ln\frac{v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H}}{\lambda_2(v_{2H}-\hat{b})} \\ &-\frac{\lambda_2(v_{1H}-\hat{b})}{(1-\lambda_1)(v_{2H}-v_{1H})}\end{aligned}$$

and

$$\begin{aligned}&(1-\lambda_1)(1-\lambda_2)(v_{1H}-v_{2H})\Pr\{1_H \text{ def } 2_H\} \\ =\ &\frac{\lambda_2(v_{1H}-\hat{b})(v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H})}{v_{1H}-v_{2H}}\ln\frac{v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H}}{\lambda_2(v_{2H}-\hat{b})} \\ &+(1-\lambda_2)\lambda_2(v_{1H}-\hat{b})\end{aligned}$$

Evaluation of R^F

$$\begin{aligned}R^F &= \lambda_2\hat{b}+(1-\lambda_2)v_{1H}+\frac{\lambda_1\lambda_2(v_{1H}-\hat{b})(v_{2L}-v_{1L})}{v_{1H}-v_{2L}}\ln\frac{(v_{2L}-\hat{b})(v_{1H}-v_{1L})}{(v_{1H}-\hat{b})(v_{2L}-v_{1L})} \\ &+\frac{\lambda_2(v_{1H}-\hat{b})(v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H})}{v_{1H}-v_{2H}}\ln\frac{v_{2H}-\lambda_2\hat{b}-(1-\lambda_2)v_{1H}}{\lambda_2(v_{2H}-\hat{b})}\end{aligned}$$

An expression for $\hat{b}$ is found by solving (2):

$$\hat{b} = \frac{1}{2\lambda_2}\left(v_{2H}+\lambda_1 v_{1L}-(1-\lambda_2)v_{1H}+(\lambda_2-\lambda_1)v_{2L}-Q\right) \tag{22}$$

with

$$Q=\sqrt{((1-\lambda_2)v_{1H}+(\lambda_1-\lambda_2)v_{2L}-\lambda_1 v_{1L}-v_{2H})^2-4\lambda_2(((1-\lambda_1)v_{2H}-(1-\lambda_2)v_{1H})v_{2L}+\lambda_1 v_{1L}v_{2H})}$$

6.3.2 The BNE of Proposition 1(ii) when $v_{1L}=v_{2L}$ (footnote 14)

$$\begin{aligned}S^F &= \lambda_1\lambda_2 v_{1L}+\lambda_1(1-\lambda_2)v_{2H}+\lambda_2(1-\lambda_1)v_{1H}+(1-\lambda_1)(1-\lambda_2)(v_{1H}+(v_{2H}-v_{1H})\Pr\{2_H \text{ def } 1_H\}) \\ U^F &= (1-\lambda_1)(v_{1H}-\lambda_2 v_L-(1-\lambda_2)v_{1H})+(1-\lambda_2)(v_{2H}-\lambda_2 v_L-(1-\lambda_2)v_{1H})\end{aligned}$$

Therefore

$$\begin{aligned}R^F &= \lambda_2(2-\lambda_2)v_{1L}-(1-\lambda_1)(1-\lambda_2)v_{2H}+(2-\lambda_1-\lambda_2)(1-\lambda_2)v_{1H} \\ &+(1-\lambda_1)(1-\lambda_2)(v_{2H}-v_{1H})\Pr\{2_H \text{ def } 1_H\}\end{aligned}$$

Derivation of $\Pr\{2_H$ **def** $1_H\}$ For the case that $v_{1H} \neq v_{2H}$ we need to evaluate

$$\begin{aligned}
\Pr\{2_H \text{ def } 1_H\} &= \int_{v_{1L}}^{\lambda_2 v_{1L}+(1-\lambda_2)v_{1H}} G'_{2H}(b)G_{1H}(b)db \\
&= \int_{v_{1L}}^{\lambda_2 v_{1L}+(1-\lambda_2)v_{1H}} \frac{\lambda_2}{1-\lambda_2}\frac{v_{1H}-v_{1L}}{(v_{1H}-b)^2}\frac{1}{1-\lambda_1}(\frac{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}}{v_{2H}-b}-\lambda_1)db \\
&= \frac{\lambda_2(v_{1H}-v_{1L})}{(1-\lambda_2)(1-\lambda_1)}\int_{v_{1L}}^{\lambda_2 v_{1L}+(1-\lambda_2)v_{1H}} \left(\frac{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}}{(v_{2H}-b)(v_{1H}-b)^2}-\frac{\lambda_1}{(v_{1H}-b)^2}\right)db
\end{aligned}$$

We exploit

$$\int \frac{1}{(v_{2H}-b)(v_{1H}-b)^2}db = \frac{1}{(v_{2H}-v_{1H})^2}\ln\left|\frac{v_{1H}-b}{v_{2H}-b}\right| + \frac{1}{(v_{2H}-v_{1H})(v_{1H}-b)}$$

to obtain

$$\begin{aligned}
&\int_{v_{1L}}^{\lambda_2 v_{1L}+(1-\lambda_2)v_{1H}} \frac{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}}{(v_{2H}-b)(v_{1H}-b)^2}db \\
=\;& \frac{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}}{(v_{2H}-v_{1H})^2}\ln\frac{\lambda_2(v_{2H}-v_{1L})}{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}} \\
&+\frac{(1-\lambda_2)(v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H})}{\lambda_2(v_{2H}-v_{1H})(v_{1H}-v_{1L})}
\end{aligned}$$

Moreover,

$$\int_{v_{1L}}^{\lambda_2 v_{1L}+(1-\lambda_2)v_{1H}} \frac{\lambda_1}{(v_{1H}-b)^2}db = \frac{\lambda_1(1-\lambda_2)}{\lambda_2(v_{1H}-v_{1L})}$$

thus

$$\begin{aligned}
\Pr\{2_H \text{ def } 1_H\} &= \frac{\lambda_2(v_{1H}-v_{1L})(v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H})}{(1-\lambda_2)(1-\lambda_1)(v_{2H}-v_{1H})^2}\ln\frac{\lambda_2(v_{2H}-v_{1L})}{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}} \\
&+\frac{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}}{(1-\lambda_1)(v_{2H}-v_{1H})}-\frac{\lambda_1}{1-\lambda_1}
\end{aligned}$$

and

$$\begin{aligned}
&(1-\lambda_1)(1-\lambda_2)(v_{2H}-v_{1H})\Pr\{2_H \text{ def } 1_H\} \\
=\;& \frac{(v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H})\lambda_2(v_{1H}-v_{1L})}{v_{2H}-v_{1H}}\ln\frac{\lambda_2(v_{2H}-v_{1L})}{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}} \\
&+(1-\lambda_2)\left((\lambda_2+\lambda_1-1)v_{1H}+(1-\lambda_1)v_{2H}-\lambda_2 v_{1L}\right)
\end{aligned}$$

Evaluation of R^F

$$\begin{aligned}
R^F &= \lambda_2 v_{1L}+(1-\lambda_2)v_{1H} \\
&+\frac{(v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H})\lambda_2(v_{1H}-v_{1L})}{v_{2H}-v_{1H}}\ln\frac{\lambda_2(v_{2H}-v_{1L})}{v_{2H}-\lambda_2 v_{1L}-(1-\lambda_2)v_{1H}}
\end{aligned} \tag{23}$$

6.3.3 The BNE in Proposition 1(iii)

$$\begin{aligned} S^F &= \lambda_1\lambda_2 v_{2L} + \lambda_1(1-\lambda_2)v_{2H} + \lambda_2(1-\lambda_1)v_{1H} + (1-\lambda_1)(1-\lambda_2)(v_{2H} + (v_{1H}-v_{2H})\Pr\{1_H \text{ def } 2_H\}) \\ U^F &= (1-\lambda_1)(v_{1H} - \lambda_1 v_{1L} - (1-\lambda_1)v_{2H}) + (1-\lambda_2)(v_{2H} - \lambda_1 v_{1L} - (1-\lambda_1)v_{2H}) + \lambda_2\lambda_1(v_{2L} - v_{1L}) \end{aligned}$$

Therefore

$$\begin{aligned} R^F &= \lambda_1(2-\lambda_1)v_{1L} - (1-\lambda_1)(1-\lambda_2)v_{1H} + (1-\lambda_1)(2-\lambda_1-\lambda_2)v_{2H} \\ &\quad +(1-\lambda_1)(1-\lambda_2)(v_{1H}-v_{2H})\Pr\{1_H \text{ def } 2_H\} \end{aligned}$$

Derivation of $\Pr\{1_H$ **def** $2_H\}$ For the case that $v_{1H} \neq v_{2H}$ we need to evaluate

$$\begin{aligned} \Pr\{1_H \text{ def } 2_H\} &= \int_{v_{1L}}^{\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}} G'_{1H}(b)G_{2H}(b)db \\ &= \int_{v_{1L}}^{\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}} \frac{\lambda_1}{1-\lambda_1}\frac{v_{2H}-v_{1L}}{(v_{2H}-b)^2}\frac{1}{1-\lambda_2}\left(\frac{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}{v_{1H}-b} - \lambda_2\right)db \\ &= \frac{\lambda_1(v_{2H}-v_{1L})}{(1-\lambda_1)(1-\lambda_2)}\int_{v_{1L}}^{\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}} \left(\frac{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}{(v_{1H}-b)(v_{2H}-b)^2} - \frac{\lambda_2}{(v_{2H}-b)^2}\right)db \end{aligned}$$

We exploit

$$\int \frac{1}{(v_{1H}-b)(v_{2H}-b)^2}db = \frac{1}{(v_{1H}-v_{2H})^2}\ln\left|\frac{v_{2H}-b}{v_{1H}-b}\right| + \frac{1}{(v_{1H}-v_{2H})(v_{2H}-b)}$$

to obtain

$$\begin{aligned} &\int_{v_{1L}}^{\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}} \frac{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}{(v_{2H}-b)^2(v_{1H}-b)}db \\ =\ & \frac{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}{(v_{1H}-v_{2H})^2}\ln\frac{\lambda_1(v_{1H}-v_{1L})}{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}} \\ & +\frac{(1-\lambda_1)(v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H})}{\lambda_1(v_{1H}-v_{2H})(v_{2H}-v_{1L})} \end{aligned}$$

Moreover,

$$\int_{v_{1L}}^{\lambda_1 v_{1L}+(1-\lambda_1)v_{2H}} \frac{\lambda_2}{(v_{2H}-b)^2}db = \frac{\lambda_2(1-\lambda_1)}{\lambda_1(v_{2H}-v_{1L})}$$

thus

$$\begin{aligned} \Pr\{1_H \text{ def } 2_H\} &= \frac{\lambda_1(v_{2H}-v_{1L})(v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H})}{(1-\lambda_1)(1-\lambda_2)(v_{1H}-v_{2H})^2}\ln\frac{\lambda_1(v_{1H}-v_{1L})}{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}} \\ &\quad +\frac{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}}{(1-\lambda_2)(v_{1H}-v_{2H})} - \frac{\lambda_2}{1-\lambda_2} \end{aligned}$$

and

$$\begin{aligned} & (1-\lambda_1)(1-\lambda_2)(v_{1H}-v_{2H})\Pr\{1_H \text{ def } 2_H\} \\ =\ & \frac{\lambda_1(v_{2H}-v_{1L})(v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H})}{v_{1H}-v_{2H}}\ln\frac{\lambda_1(v_{1H}-v_{1L})}{v_{1H}-\lambda_1 v_{1L}-(1-\lambda_1)v_{2H}} \\ & +(1-\lambda_1)((1-\lambda_2)v_{1H}-\lambda_1 v_{1L}+(\lambda_1+\lambda_2-1)v_{2H}) \end{aligned}$$

Evaluation of R^F

$$R^F = \lambda_1 v_{1L} + (1-\lambda_1)v_{2H} + \frac{(v_{1H} - \lambda_1 v_{1L} - (1-\lambda_1)v_{2H})\lambda_1(v_{2H} - v_{1L})}{v_{1H} - v_{2H}} \ln \frac{\lambda_1 (v_{1H} - v_{1L})}{v_{1H} - \lambda_1 v_{1L} - (1-\lambda_1)v_{2H}}$$

7 Proof of Lemma 1

Given $\lambda_1 = \lambda_2$ and $v_{1L} = v_{2L} = v_L$, when $v_{1H} < v_{2H} = v_H$ Proposition 1(ii) (footnote 14) applies and reveals that types 1_L,2_L bid as in the benchmark symmetric setting, whereas $G_{1H}(b) = \frac{1}{1-\lambda}(\frac{v_H - \lambda v_L - (1-\lambda)v_{1H}}{v_H - b} - \lambda)$ and $G_{2H}(b) = \frac{\lambda}{1-\lambda}\frac{b-v_L}{v_{1H}-b}$ with support $[v_L, \bar{b}]$, in which $\bar{b} = \lambda v_L + (1-\lambda)v_{1H}$. It is simple to see that both $G_{1H}(b)$ and $G_{2H}(b)$ are decreasing with respect to v_{1H} for any $b \in (v_L, \bar{b})$, and this implies that 1_H and 2_H are both more aggressive, in the sense of first order stochastic dominance, the larger is v_{1H} in $(v_L, v_H]$.[39] Given that

$$R^F = \lambda^2 v_L + \lambda(1-\lambda)\int_{v_L}^{\bar{b}} bdG_{2H}(b) + \lambda(1-\lambda)\int_{v_L}^{\bar{b}} bdG_{1H}(b) + (1-\lambda)^2 \int_{v_L}^{\bar{b}} bd(G_{1H}(b)G_{2H}(b)) \quad (24)$$

we infer that R^F is increasing in v_{1H}.

When $v_{1H} > v_H$, Proposition 1(iii) applies and reveals that types $1_L, 1_H, 2_L$ bid as in the benchmark symmetric setting, whereas $G_{2H}(b) = \frac{(1-\lambda)(v_{1H}-v_H)+\lambda(b-v_L)}{(1-\lambda)(v_{1H}-b)}$ for any $b \in [v_L, \lambda v_L + (1-\lambda)v_H]$. Since $G_{2H}(b)$ is strictly increasing in v_{1H} for any $b \in [v_L, \lambda v_L + (1-\lambda)v_H)$, we infer that 2_H is less aggressive, in the sense of first order stochastic dominance, the larger is v_{1H}. Using again (24), after replacing G_{1H} with G_H and $\bar{b}$ with $\lambda v_L + (1-\lambda)v_H$, it follows that R^F is strictly decreasing with respect to v_{1H}.

8 Proof of Proposition 4

8.1 Proof of Proposition 4(i)

8.1.1 The proof when (9) or (10) is satisfied

The proofs for these results are provided in the text.

8.1.2 The proof when (11) is satisfied

Since $R^S > R^F$ when (9) is satisfied and R^S and R^F are continuous functions of the valuations, it follows that $R^S > R^F$ if $v_{1L} \leq v_{2L} \leq v_{1H} < v_{2H}$ and v_{2L} is close to v_{1L}.

8.1.3 The proof when (12) is satisfied

If $\lambda \geq \frac{1}{2}$, then the condition $\lambda_2 \geq \max\{\frac{1}{2}, \lambda_1\}$ for Proposition 5(iia) is satisfied. Hence the proof in Proposition 5(iia) applies to this setting to show that $R^S > R^F$ for each profile of valuations in region B, that is such that $v_{1L} \leq v_{2L} \leq v_{1H} < v_{2H}$.

[39] Precisely, if $v_{1H} < v'_{1H} < v_H$, then F_{1H} and F_{2H} given v'_{1H} first order stocastically dominate, respectively, F_{1H} and F_{2H} given v_{1H}.

For valuations in region C, that is $v_{1L} < v_{1H} < v_{2L} < v_{2H}$, we show that $U^F \geq U^S$, and thus $R^S > R^F$, if (12) is satisfied. Since $v_{1H} < v_{2H}$, Proposition 1(ii) applies and thus the aggregate bidders' rents in the FPA are $U^F = (1-\lambda)(v_{1H} - \bar{b}) + (1-\lambda)(v_{2H} - \bar{b}) + \lambda^2(v_{2L} - v_{1L})$ with $\bar{b} = \lambda\hat{b} + (1-\lambda)v_{1H}$. Since $U^S = \lambda v_{2L} + (1-\lambda)v_{2H} - \lambda v_{1L} - (1-\lambda)v_{1H}$, the difference $U^F - U^S$ is equal to $\lambda(1-\lambda)(v_{1L} + 2v_{1H} - v_{2L} - 2\hat{b})$. From (22) we obtain $\hat{b} = \frac{1}{2\lambda}(\lambda v_{1L} + v_{2H} - (1-\lambda)v_{1H} - Q)$ with $Q = \sqrt{((1-\lambda)v_{1H} - \lambda v_{1L} - v_{2H})^2 - 4\lambda(1-\lambda)(v_{2H} - v_{1H})v_{2L} - 4\lambda^2 v_{1L}v_{2H}}$. Therefore $U^F \geq U^S$ boils down to $Q \geq v_{2H} + \lambda v_{2L} - (1+\lambda)v_{1H}$ and (after squaring – notice that $v_{2H} + \lambda v_{2L} - (1+\lambda)v_{1H} > 0$) ultimately to

$$\begin{aligned} &-\lambda v_{2L}^2 + 2(3v_{1H} - 3v_{2H} + 2\lambda v_{2H} - \lambda v_{1H})v_{2L} \\ &+\lambda v_{1L}^2 - 4v_{1H}^2 + 4v_{1H}v_{2H} + 2(1-2\lambda)v_{1L}v_{2H} - 2(1-\lambda)v_{1H}v_{1L} \geq 0 \end{aligned} \tag{25}$$

We prove that this inequality holds for each $v_{2L} \in (v_{1H}, v_{1H} + \frac{2\lambda-1}{3-2\lambda}(v_{1H} - v_{1L})$ by verifying that the left hand side of (25) is positive both at $v_{2L} = v_{1H}$ and at $v_{2L} = v_{1H} + \frac{2\lambda-1}{3-2\lambda}(v_{1H} - v_{1L})$. At $v_{2L} = v_{1H}$, the left hand side in (25) reduces to $(v_{1H} - v_{1L})[\lambda(4v_{2H} - 3v_{1H} - v_{1L}) - 2(v_{2H} - v_{1H})]$ which is positive since (i) it is increasing in λ; (ii) has value $\frac{1}{2}(v_{1H} - v_{1L})^2 > 0$ at $\lambda = \frac{1}{2}$. At $v_{2L} = v_{1H} + \frac{2\lambda-1}{3-2\lambda}(v_{1H} - v_{1L})$, the left hand side in (25) reduces to $\frac{8(1-\lambda)}{(3-2\lambda)^2}(v_{1H} - v_{1L})^2 > 0$.

8.2 Proof of Proposition 4(ii)

Given that $\lambda_1 = \lambda_2$, the condition $\lambda_2 \geq \lambda_1$ for Proposition 5(iib) is satisfied. Hence the proof in Proposition 5(iib) applies to this setting to show that $R^F - R^S$ is increasing with respect to v_{2L} in region C.

Proof for the case of distribution shift In the case of shift, $v_{2H} - v_{1H} = \alpha$ and $v_{2L} - v_{1L} = \alpha$. If $\alpha \leq v_{1H} - v_{1L}$, then $v_{2L} \leq v_{1H}$ and $U^S = \lambda^2(v_{2L} - v_{1L}) + \lambda(1-\lambda)(v_{2H} - v_{1L}) + (1-\lambda)\lambda(v_{1H} - v_{2L}) + (1-\lambda)^2(v_{2H} - v_{1H}) = (1 - 2\lambda + 2\lambda^2)\alpha + 2\lambda(1-\lambda)(v_{1H} - v_{1L})$. As a consequence, $U^F \geq U^S$ reduces to $2\lambda(v_{1H} - v_{1L}) \geq (2 - 3\lambda)\alpha$. If $\lambda > \frac{2}{5}$, then this inequality is satisfied for any $\alpha \leq v_{1H} - v_{1L}$; if instead $\lambda \leq \frac{2}{5}$, then the inequality is violated for $\alpha = v_{1H} - v_{1L}$ and it holds if and only if $\alpha \leq \frac{2\lambda}{2-3\lambda}(v_{1H} - v_{1L})$.

If $\alpha > v_{1H} - v_{1L}$, then $v_{2L} > v_{1H}$ and $U^S = \lambda^2(v_{2L} - v_{1L}) + \lambda(1-\lambda)(v_{2H} - v_{1L}) + \lambda(1-\lambda)(v_{2L} - v_{1H}) + (1-\lambda)^2(v_{2H} - v_{1H}) = \alpha$. As a consequence, $U^F \geq U^S$ reduces to $2(2+\lambda)(v_{1H} - v_{1L}) \geq 3(2-\lambda)\alpha$. If order for this inequality to be satisfied by an α larger than $v_{1H} - v_{1L}$ it is necessary that $\lambda > \frac{2}{5}$.

9 Proof of the claims in Subsection 4.2.4

When (3) is satisfied, $G_2(b) \leq G_1(b)$ holds for any b. Moreover, bidder 1 never wins in either auction when (3) holds. Conversely, 2 wins with probability one and in the FPA he pays v_{1H}; in the SPA his expected payment is the expected valuation of bidder 1, which is smaller than v_{1H}.

For $i = 1, 2$, let U_i^F denote bidder i's ex ante expected equilibrium payoff in the FPA; U_i^S is defined likewise for the SPA. When (4) holds we find $U_1^F = (1-\lambda)\lambda(v_{1H} - \hat{b})$, $U_1^S = (1-\lambda)\lambda \max\{v_{1H} - v_{2L}, 0\}$, and $U_1^F > U_1^S$ since $\hat{b} < \min\{v_{2L}, v_{1H}\}$. Moreover, $U_2^F = \lambda^2(v_{2L} - v_{1L}) + (1-\lambda)[v_{2H} - \lambda\hat{b} - (1-\lambda)v_{1H}]$, $U_2^S = \lambda[\lambda(v_{2L} - v_{1L}) + (1-\lambda)\max\{v_{2L} - v_{1H}, 0\}] + (1-\lambda)[v_{2H} - \lambda v_{1L} - (1-\lambda)v_{1H}]$, and $U_2^S - U_2^F = (1-\lambda)\lambda[\max\{v_{2L} - v_{1H}, 0\} + \hat{b} - v_{1L}] > 0$ since $\hat{b} > v_{1L}$. For the equilibrium bid distributions we find $G_1(b) > G_2(b)$ for any $b \in [v_{1L}, \hat{b}]$ as $G_1(v_{1L}) = G_2(\hat{b}) = \lambda$. For $b \in (\hat{b}, \bar{b}]$, $G_1(b) = \frac{v_{2H} - \bar{b}}{v_{2H} - b}$ and $G_2(b) = \frac{v_{1H} - \bar{b}}{v_{1H} - b}$, hence $G_1(b) > G_2(b)$ for $b \in (\hat{b}, \bar{b})$.

When (7) holds we obtain $U_1^F = (1-\lambda)(v_{1H} - \lambda v_{1L} - (1-\lambda)v_{2H})$, $U_1^S = (1-\lambda)(v_{1H} - \lambda v_{2L} - (1-\lambda)v_{2H})$, and $U_1^F \geq U_1^S$ since $v_{1L} \leq v_{2L}$. Moreover, $U_2^F = U_2^S = \lambda^2(v_{2L} - v_{1L}) + (1-\lambda)\lambda(v_{2H} - v_{1L})$. For the equilibrium bid distributions we find $G_1(b) = \lambda\frac{v_{2H} - v_{1L}}{v_{2H} - b}$ and $G_2(b) = \frac{v_{1H} - \bar{b}}{v_{1H} - b}$ with $\bar{b} = \lambda v_{1L} + (1-\lambda)v_{2H}$ and $G_2(b) > G_1(b)$ for any $b \in [v_{1L}, \bar{b})$.

10 Proof of the final claim in Subsection 4.2.5

We consider two sequences of atomless c.d.f. $\{F_1^n, F_2^n\}_{n=1}^{+\infty}$, with continuous and positive densities f_1^n, f_2^n for each n, which converges weakly to $\tilde{F}_1, \tilde{F}_2$. We show that for any large n, (13) and/or (14) are violated by F_1^n, F_2^n.

When $v_{1L} < v_{2L}$, select an arbitrary $\hat{v} \in (v_{1L}, v_{2L})$ and notice that given a small $\varepsilon > 0$, for a large n the inequality $F_1^n(\hat{v}) > \lambda - \varepsilon$ holds. Therefore $r^n(\hat{v}) = (F_2^n)^{-1}[F_1^n(\hat{v})] \geq v_{2L} - \varepsilon > \hat{v}$ [because $\lim_{n\to+\infty} F_2^n(v) = 0$ for each $v < v_{2L} - \varepsilon$] and $\int_{\hat{v}}^{r^n(\hat{v})} f_2^n(x)dx = F_2^n[r^n(\hat{v})] - F_2^n(\hat{v}) > \lambda - 2\varepsilon$ for a large n. If $f_1^n(\hat{v}) \geq f_2^n(x)$ for any $x \in [\hat{v}, r^n(\hat{v})]$, then $\lim_{n\to+\infty} f_1^n(\hat{v}) = 0$ implies $\lim_{n\to+\infty} \int_{\hat{v}}^{r^n(\hat{v})} f_2^n(x)dx = 0$: contradiction. Hence (14) is violated if F_1^n, F_2^n are close to $\tilde{F}_1, \tilde{F}_2$ and $v_{1L} < v_{2L}$.

Now assume that $v_{1L} = v_{2L}$ and $v_{1H} < v_{2H}$. Then given a small $\varepsilon > 0$ and a large n, the inequality $F_1^n(v_{1H} + \varepsilon) - F_1^n(v_{1H} - \varepsilon) = \int_{v_{1H}-\varepsilon}^{v_{1H}+\varepsilon} f_1^n(x)dx > 1 - \lambda - \varepsilon$ holds, and $F_2^n(v_{1H} + \varepsilon) - F_2^n(v_{1H} - \varepsilon) = \int_{v_{1H}-\varepsilon}^{v_{1H}+\varepsilon} f_2^n(x)dx$ tends to zero. Now notice that if there exists a number $t > 0$ such that $\frac{f_1^n(x)}{f_2^n(x)} \leq t$ for any $x \in (v_{1H} - \varepsilon, v_{1H} + \varepsilon)$ and any n, then $\int_{v_{1H}-\varepsilon}^{v_{1H}+\varepsilon} f_1^n(x)dx \leq t \int_{v_{1H}-\varepsilon}^{v_{1H}+\varepsilon} f_2^n(x)dx$ and $\lim_{n\to+\infty} \int_{v_{1H}-\varepsilon}^{v_{1H}+\varepsilon} f_1^n(x)dx = 0$. Thus for any $t > 0$, for any large n there exists some $x_n \in (v_{1H} - \varepsilon, v_{1H} + \varepsilon)$ such that $\frac{f_1^n(x_n)}{f_2^n(x_n)} > t$, which implies that (13) cannot hold since $F_2^n(x_n) > \lambda - \varepsilon$.

11 Proof of Proposition 5

(i) Suppose that $\lambda_1 < \lambda_2$. Then Proposition 1(ii) applies and the ex ante expected payoffs of bidders 1 and 2 in the FPA and in the SPA are

$$U_1^F = (1-\lambda_1)\lambda_2(v_H - \hat{b}) \quad \text{and} \quad U_1^S = (1-\lambda_1)\lambda_2(v_H - v_{2L})$$
$$U_2^F = \lambda_2\lambda_1(v_{2L} - v_{1L}) + (1-\lambda_2)\lambda_2(v_H - \hat{b}) \quad \text{and} \quad U_2^S = \lambda_2\lambda_1(v_{2L} - v_{1L}) + (1-\lambda_2)\lambda_1(v_H - v_{1L})$$

From (2) we obtain $\hat{b} = v_{2L} - \frac{\lambda_1}{\lambda_2}(v_{2L} - v_{1L})$, and this reveals that $U_1^F > U_1^S$ and $U_2^F > U_2^S$.

In the opposite case such that $\lambda_1 \geq \lambda_2$, Proposition 1(iii) applies and

$$\begin{aligned} U_1^F &= (1-\lambda_1)\lambda_1(v_H - v_{1L}) > U_1^S = (1-\lambda_1)\lambda_2(v_H - v_{2L}) \\ U_2^F &= U_2^S = \lambda_2\lambda_1(v_{2L} - v_{1L}) + (1-\lambda_2)\,\lambda_1(v_H - v_{1L}) \end{aligned}$$

In either case, $U^F = U_1^F + U_2^F > U^S = U_1^S + U_2^S$ and thus $R^S > R^F$.

(iia) Since $\lambda_2 \geq \lambda_1$, inequality (4) holds in region B and Proposition 1(ii) applies for the FPA. First we notice that for $v_{2L} = v_{1L}$, R^F is decreasing in v_{2H}. It suffices to notice from footnote 14 that an increase in v_{2H} has the only effect of making 1_H less aggressive by increasing $G_{1H}(b)$. However, an increase in v_{2H} does not affect R^S. Since $R^S > R^F$ at $v_{2H} = v_{1H}$, it follows that $R^S > R^F$ still holds for $v_{2H} > v_{1H}$. As a consequence, $R^S > R^F$ in region B if v_{2L} is close to v_{1L}. Now we show that $R^S > R^F$ in region B if $\lambda_2 \geq \max\{\frac{1}{2}, \lambda_1\}$ by proving that $U^F \geq U^S$ for any profile of values in B. The bidders' rents in the FPA are $U^F = (1-\lambda_1)(v_{1H} - \bar{b}) + \lambda_2\lambda_1(v_{2L} - v_{1L}) + (1-\lambda_2)(v_{2H} - \bar{b})$ with $\bar{b} = \lambda_2\hat{b} + (1-\lambda_2)v_{1H}$. On the other hand, the bidders' rents in the SPA are $U^S = \lambda_1\lambda_2(v_{2L} - v_{1L}) + \lambda_1(1-\lambda_2)(v_{2H} - v_{1L}) + (1-\lambda_1)\lambda_2(v_{1H} - v_{2L}) + (1-\lambda_1)(1-\lambda_2)(v_{2H} - v_{1H})$. Hence the inequality $U^F \geq U^S$ reduces to

$$(\lambda_2 - \lambda_1)(1-\lambda_2)v_{1H} + \lambda_1(1-\lambda_2)v_{1L} + \lambda_2(1-\lambda_1)v_{2L} \geq \lambda_2(2 - \lambda_1 - \lambda_2)\hat{b} \tag{26}$$

We show that (26) holds in region B if $\lambda_2 \geq \max\{\frac{1}{2}, \lambda_1\}$. First we notice that (26) depends on v_{2H} only through $\hat{b}$, and we prove that $\hat{b}$ is (weakly) increasing with respect to v_{2H}. Precisely, we use Z to denote the left hand side in (2), thus $\frac{\partial \hat{b}}{\partial v_{2H}} = -\frac{\frac{\partial Z}{\partial v_{2H}}\big|_{b=\hat{b}}}{\frac{\partial Z}{\partial b}\big|_{b=\hat{b}}}$. Since $\hat{b}$ is the smallest solution of (2), it follows that $\frac{\partial Z}{\partial b}\big|_{b=\hat{b}} < 0$. Moreover, $\frac{\partial Z}{\partial v_{2H}}\Big|_{b=\hat{b}} = \lambda_1 v_{1L} + (1-\lambda_1)v_{2L} - \hat{b}$ and $\hat{b} \leq \lambda_1 v_{1L} + (1-\lambda_1)v_{2L}$ since Z evaluated at $b = \lambda_1 v_{1L} + (1-\lambda_1)v_{2L}$ is equal to $-\lambda_1(1-\lambda_2)(v_{2L} - v_{1L})[v_{1H} - \lambda_1 v_{1L} - (1-\lambda_1)v_{2L}] \leq 0$. Therefore $\frac{\partial Z}{\partial v_{2H}}\Big|_{b=\hat{b}} > 0$ and $\frac{\partial \hat{b}}{\partial v_{2H}} > 0$. Using (22) we see that $\lim_{v_{2H}\to+\infty} \hat{b} = \lambda_1 v_{1L} + (1-\lambda_1)v_{2L}$, hence a sufficient condition for (26) to hold is $(\lambda_2 - \lambda_1)(1-\lambda_2)v_{1H} + \lambda_1(1-\lambda_2)v_{1L} + \lambda_2(1-\lambda_1)v_{2L} \geq \lambda_2(2-\lambda_1-\lambda_2)(\lambda_1 v_{1L} + (1-\lambda_1)v_{2L})$, which is equivalent to

$$(\lambda_2 - \lambda_1)(1-\lambda_2)v_{1H} + \lambda_1(1 - 3\lambda_2 + \lambda_1\lambda_2 + \lambda_2^2)v_{1L} + \lambda_2(1-\lambda_1)(\lambda_1 + \lambda_2 - 1)v_{2L} \geq 0 \tag{27}$$

Since the left hand side in (27) is linear in v_{2L} and $v_{1L} \leq v_{2L} \leq v_{1H}$ in region B, we deduce that (27) holds in region B if and only if it is satisfied at $v_{2L} = v_{1L}$ and at $v_{2L} = v_{1H}$. At $v_{2L} = v_{1L}$, (27) reduces to $(1-\lambda_2)(\lambda_2 - \lambda_1)(v_{1H} - v_{1L}) \geq 0$, which holds as $\lambda_2 > \lambda_1$. At $v_{2L} = v_{1H}$, (27) reduces to $\lambda_1(3\lambda_2 - \lambda_1\lambda_2 - \lambda_2^2 - 1)(v_{1H} - v_{1L}) \geq 0$, which holds as (i) the left hand side is increasing in λ_2; (ii) if $\lambda_1 < \frac{1}{2}$, then $\lambda_2 \geq \max\{\frac{1}{2}, \lambda_1\} = \frac{1}{2}$ implies $3\lambda_2 - \lambda_1\lambda_2 - \lambda_2^2 - 1 \geq \frac{1}{4} - \frac{1}{2}\lambda_1 > 0$; (ii) if $\lambda_1 \geq \frac{1}{2}$, then $\lambda_2 \geq \max\{\frac{1}{2}, \lambda_1\} = \lambda_1$ implies $3\lambda_2 - \lambda_1\lambda_2 - \lambda_2^2 - 1 \geq 3\lambda_1 - 2\lambda_1^2 - 1 = (1-\lambda_1)(2\lambda_1 - 1) \geq 0$.

Now consider region C, that is valuations such that $v_{1L} < v_{1H} < v_{2L} < v_{2H}$. Then $U^F = (1-\lambda_1)\lambda_2(v_{1H} - \hat{b}) + \lambda_2\lambda_1(v_{2L} - v_{1L}) + (1-\lambda_2)(v_{2H} - \lambda_2\hat{b} - (1-\lambda_2)v_{1H})$ with $\bar{b} = \lambda\hat{b} + (1-\lambda)v_{1H}$, and $U^S = \lambda_2 v_{2L} + (1-\lambda_2)v_{2H} - \lambda_1 v_{1L} - (1-\lambda_1)v_{1H}$. The inequality $U^F \geq U^S$ is equivalent to $-\lambda_2(1-\lambda_1)v_{2L}+\lambda_1(1-\lambda_2)v_{1L}+(3\lambda_2-\lambda_2\lambda_1-\lambda_2^2-\lambda_1)v_H \geq \lambda_2(2-\lambda_2-\lambda_1)\hat{b}$. Using (22) we obtain that $U^F \geq U^S$ boils down to a inequality which is quadratic in v_{2L}, with (complicated) coefficients: $-4\lambda_2(1-\lambda_1)(3\lambda_2+\lambda_1^2-\lambda_2^2-\lambda_1\lambda_2-2\lambda_1)$ for v_{2L}^2, $-4\lambda_2(1-\lambda_1)(3-\lambda_1-\lambda_2)(2-\lambda_1-\lambda_2)$ $v_{2H}+4(6v_{1H}\lambda_2-2v_{1L}\lambda_1^2+v_{1L}\lambda_1^3+2v_{1H}\lambda_1^2-v_{1H}\lambda_1^3+v_{1H}\lambda_2^2-v_{1H}\lambda_2^3-4v_{1L}\lambda_1\lambda_2^2+3v_{1L}\lambda_1^2\lambda_2+v_{1L}\lambda_1\lambda_2^3-$ $2v_{1L}\lambda_1^3\lambda_2+3v_{1H}\lambda_1\lambda_2^2+9v_{1H}\lambda_1^2\lambda_2-v_{1H}\lambda_1^3\lambda_2+v_{1L}\lambda_1^2\lambda_2^2-v_{1H}\lambda_1^2\lambda_2^2+2v_{1L}\lambda_1\lambda_2-17v_{1H}\lambda_1\lambda_2)$ for v_{2L}, and $4(2-\lambda_1-\lambda_2)\left((\lambda_1+\lambda_1^2\lambda_2-3\lambda_2\lambda_1+\lambda_1\lambda_2^2)v_{1L}+(3\lambda_2-\lambda_2\lambda_1-\lambda_2^2-\lambda_1)v_{1H}\right)v_{2H}+4(1-\lambda_1)(2v_{1H}-v_{1L}\lambda_1)\left(-\lambda_1(1-\lambda_2)v_{1L}+(-3\lambda_2+\lambda_2\lambda_1+\lambda_2^2+\lambda_1)v_{1H}\right)$ as a constant term.

We prove that the inequality is satisfied if $v_{1H} < v_{2L} \leq v_{1H}+\frac{\lambda_1(3\lambda_2-\lambda_2^2-\lambda_1\lambda_2-1)}{\lambda_2(1-\lambda_1)(3-\lambda_1-\lambda_2)}(v_{1H}-v_{1L})$. In order to do so, we notice that the coefficient of v_{2L}^2 is negative, that is $3\lambda_2+\lambda_1^2-\lambda_2^2-\lambda_1\lambda_2-2\lambda_1 > 0$, and thus it suffices to verify that the inequality holds at $v_{2L}=v_{1H}$ and at $v_{2L}=v_{1H}+\frac{\lambda_1(3\lambda_2-\lambda_2^2-\lambda_1\lambda_2-1)}{\lambda_2(1-\lambda_1)(3-\lambda_1-\lambda_2)}(v_{1H}-v_{1L})$. In particular, $3\lambda_2+\lambda_1^2-\lambda_2^2-\lambda_1\lambda_2-2\lambda_1$ is increasing in λ_2, and (i) if $\lambda_1 < \frac{1}{2}$, then $\lambda_2 \geq \frac{1}{2}$ and $3\lambda_2+\lambda_1^2-\lambda_2^2-\lambda_1\lambda_2-2\lambda_1 \geq \frac{5}{4}-\frac{5}{2}\lambda_1+\lambda_1^2 \geq \lambda_1^2$; (ii) if $\lambda_1 \geq \frac{1}{2}$, then $\lambda_2 \geq \lambda_1$ and $3\lambda_2+\lambda_1^2-\lambda_2^2-\lambda_1\lambda_2-2\lambda_1 \geq \lambda_1(1-\lambda_1) \geq 0$. At $v_{2L}=v_{1H}$, the inequality reduces to $\lambda_1(v_{1H}-v_{1L})((3\lambda_2-\lambda_2^2-\lambda_1\lambda_2-1)(2-\lambda_1-\lambda_2)v_{2H}+2v_{1H}-v_{1L}\lambda_1-7v_{1H}\lambda_2+v_{1L}\lambda_1^2-v_{1H}\lambda_1^2+5v_{1H}\lambda_2^2-v_{1H}\lambda_2^3-v_{1L}\lambda_1^2\lambda_2-2v_{1H}\lambda_1\lambda_2^2+v_{1L}\lambda_1\lambda_2+4v_{1H}\lambda_1\lambda_2) \geq 0$, and since $v_{2H} \geq v_{1H}$, the left hand side is larger than $\lambda_1(v_{1H}-v_{1L})((3\lambda_2-\lambda_2^2-\lambda_1\lambda_2-1)(2-\lambda_1-\lambda_2)v_{1H}+2v_{1H}-v_{1L}\lambda_1-7v_{1H}\lambda_2+v_{1L}\lambda_1^2-v_{1H}\lambda_1^2+5v_{1H}\lambda_2^2-v_{1H}\lambda_2^3-v_{1L}\lambda_1^2\lambda_2-2v_{1H}\lambda_1\lambda_2^2+v_{1L}\lambda_1\lambda_2+4v_{1H}\lambda_1\lambda_2)$, which is equal to $\lambda_1^2(1-\lambda_1)(1-\lambda_2)(v_{1H}-v_{1L})^2 > 0$. At $v_{2L}=v_{1H}+\frac{\lambda_1(3\lambda_2-\lambda_2^2-\lambda_1\lambda_2-1)}{\lambda_2(1-\lambda_1)(3-\lambda_1-\lambda_2)}(v_{1H}-v_{1L})$, the inequality reduces to $\lambda_1^2(1-\lambda_2)(3\lambda_2-\lambda_1-\lambda_2^2-\lambda_1\lambda_2)\frac{(2-\lambda_1-\lambda_2)^2(v_{1H}-v_{1L})^2}{\lambda_2(1-\lambda_1)(3-\lambda_1-\lambda_2)^2}$, which is positive.

(iib) Since R^S does not depend on v_{2L} in region C, we need to prove that $\frac{\partial R^F}{\partial v_{2L}} > 0$. To this purpose we notice that (4) is satisfied in region C and we show that $\frac{\partial \hat{b}}{\partial v_{2L}} > 0$. This implies $\frac{\partial \bar{b}}{\partial v_{2L}} > 0$ and from (5)-(6) it follows that $G_{1H}(b), G_{2L}(b), G_{2H}(b)$ are all decreasing in v_{2L}, which implies that types $1_H, 2_L, 2_H$ are all more aggressive as v_{2L} increases. Thus R^F is increasing with respect to v_{2L}. In order to see that $\frac{\partial \hat{b}}{\partial v_{2L}} = -\frac{\frac{\partial Z}{\partial v_{2L}}\big|_{b=\hat{b}}}{\frac{\partial Z}{\partial b}\big|_{b=\hat{b}}} > 0$, recall from the proof of Proposition 5(iia) that $\frac{\partial Z}{\partial b}\big|_{b=\hat{b}} < 0$ and $\frac{\partial Z}{\partial v_{2L}}\Big|_{b=\hat{b}} = (\lambda_1-\lambda_2)\hat{b}+(1-\lambda_1)v_{2H}-(1-\lambda_2)v_{1H}$. Since $v_{2H} > v_{1H}$ in region C, we find $\frac{\partial Z}{\partial v_{2L}}\Big|_{b=\hat{b}} > (\lambda_2-\lambda_1)(v_{1H}-\hat{b}) \geq 0$; therefore $\frac{\partial \hat{b}}{\partial v_{2L}} > 0$.

(iiia) Given that $\lambda_1 \geq \lambda_2$, in region A the inequality (7) is satisfied and thus Proposition 1(iii) applies for the FPA. This implies that G_1, G_2, the equilibrium bid distributions of the two bidders, are independent of λ_2: using (8) we find $G_1(b)=\lambda_1+(1-\lambda_1)G_{1H}(b)=\frac{\lambda_1(v_{2H}-v_{1L})}{v_{2H}-b}$ and $G_2(b)=\lambda_2+(1-\lambda_2)G_{2H}(b)=\frac{v_{1H}-(1-\lambda_1)v_{2H}-\lambda_1 v_{1L}}{v_{1H}-b}$ for $b \in [v_{1L}, \lambda_1 v_{1L}+(1-\lambda_1)v_{2H}]$. Hence R^F is independent of λ_2, whereas $R^S=\lambda_1 v_{1L}+(1-\lambda_1)(\lambda_2 v_{2L}+(1-\lambda_2)v_{2H})$ in region A, and thus R^S is decreasing in λ_2. Therefore, given $\lambda_2 \leq \lambda_1$, the minimum of R^S with respect to λ_2 is reached at $\lambda_2=\lambda_1$. Then we can apply Proposition 4 [condition (10)] to conclude that $R^S > R^F$.

(iiib) The proof is organized in four steps

Step 1: In region B, $R^F - R^S$ is increasing with respect to v_{2H} if (7) is satisfied.

In region B, R^S is independent of v_{2H}. On the other hand, Proposition 1(iii) reveals that R^F is increasing in v_{2H}: the bidding behavior of types $1_L, 2_L$ does not depend on v_{2H} whereas types

$1_H, 2_H$ bid more aggressively as v_{2H} increases [as $G_{1H}(b)$ and $G_{2H}(b)$ are decreasing in v_{2H}]. Hence $R^F - R^S$ is increasing in v_{2H}.

Step 2: In region B, $R^F > R^S$ if (4) is satisfied and $\lambda_1 > \lambda_2(1 + \ln \frac{1}{\lambda_2})$.

We start by proving that $\hat{b}$ and $\bar{b}$ are increasing with respect to v_{2L}, and then show that also R^F is increasing in v_{2L}. Precisely, we use Z to denote the left hand side in (2) and prove that $\frac{\partial \hat{b}}{\partial v_{2L}} = -\frac{\left.\frac{\partial Z}{\partial v_{2L}}\right|_{b=\hat{b}}}{\left.\frac{\partial Z}{\partial b}\right|_{b=\hat{b}}} > 0$. Since $\hat{b}$ is the smallest solution of (2), it follows that $\left.\frac{\partial Z}{\partial b}\right|_{b=\hat{b}} < 0$. Moreover, $\left.\frac{\partial Z}{\partial v_{2L}}\right|_{b=\hat{b}} = (\lambda_1 - \lambda_2)\hat{b} + (1-\lambda_1)v_{2H} - (1-\lambda_2)v_{1H}$ and (4) implies $(1-\lambda_1)v_{2H} > (1-\lambda_2)v_{1H} + (\lambda_2 - \lambda_1)v_{1L}$. Therefore $\left.\frac{\partial Z}{\partial v_{2L}}\right|_{b=\hat{b}} > (\lambda_1 - \lambda_2)\hat{b} + (1-\lambda_2)v_{1H} + (\lambda_2 - \lambda_1)v_{1L} - (1-\lambda_2)v_{1H} = (\lambda_1 - \lambda_2)(\hat{b} - v_{1L}) > 0$, and hence $\frac{\partial \hat{b}}{\partial v_{2L}} > 0$, $\frac{\partial \bar{b}}{\partial v_{2L}} > 0$.

From (5)-(6) we see that types $1_H, 2_L, 2_H$ are all more aggressive as v_{2L} increases, as in the proof of Proposition 5(iib). Thus R^F is increasing with respect to v_{2L}, and let $R^F_{\min}$ denote R^F when v_{2L} takes on its minimum value, that is at $v_{2L} = v_{1L}$. Also R^S is increasing with respect to v_{2L}, and $R^S = \lambda_1 v_{1L} + (1-\lambda_1)v_{1H}$ when v_{2L} takes on its maximum value in region B, that is at $v_{2L} = v_{1H}$. We prove below that $R^F_{\min} > \lambda_1 v_{1L} + (1-\lambda_1)v_{1H}$, which implies that $R^F > R^S$ in region B when (4) is satisfied.

When $v_{2L} = v_{1L}$, the equilibrium bidding in the FPA is described in footnote 14 and it is clear that $R^F_{\min}$ is decreasing in v_{2H}, as seen in the proof of Proposition 5(iia). Hence $R^F_{\min} > \lim_{v_{2H}\to+\infty} R^F_{\min}$, and using (23) we see that $\lim_{v_{2H}\to+\infty} R^F_{\min} = \lambda_2 v_{1L} + (1-\lambda_2)v_{1H} + \lambda_2(v_{1H} - v_{1L})\ln\lambda_2$. The inequality $\lambda_2 v_{1L} + (1-\lambda_2)v_{1H} + \lambda_2(v_{1H} - v_{1L})\ln\lambda_2 > \lambda_1 v_{1L} + (1-\lambda_1)v_{1H}$ is equivalent to $\lambda_1 > \lambda_2(1 + \ln\frac{1}{\lambda_2})$, which holds by assumption.

Step 3: If $v_{2L} \leq v_{1H}$, then there exists v^*_{2H} [and $v^*_{2H} > v_{1H}$, such that (7) is satisfied] such that $R^S > R^F$ when $v_{2H} < v^*_{2H}$, and $R^F > R^S$ when $v_{2H} > v^*_{2H}$.

This is immediate consequence of $R^S > R^F$ if $v_{2H} = v_{1H}$ [from Proposition 5(ia)], and Steps 1 and 2 in this proof.

Step 4: If $v_{2L} > v_{1H}$ is not too larger than v_{1H}, then there exists v^*_{2H} (and $v^*_{2H} > v_{1H}$) such that $R^S > R^F$ when $v_{2H} \in (v_{2L}, v^*_{2H})$, but $R^F > R^S$ when $v_{2H} > v^*_{2H}$. If conversely v_{2L} is much larger than v_{1H}, then $R^F > R^S$ for any $v_{2H} > v_{2L}$.

We start from a profile of valuations $(v_{1L}, v_{1H}, v_{2L}, v_{2H})$ such that $v_{2L} = v_{1H}$ and (7) is satisfied, and consider increasing v_{2L}, which implies that region C is entered. The increase in v_{2L} has no effect on R^F and has no effect on R^S, thus $R^F > R^S$ if and only if v_{2H} is sufficiently large.

Now start from $(v_{1L}, v_{1H}, v_{2L}, v_{2H})$ such that $v_{2L} = v_{1H}$ and (4) is satisfied. We know from Step 2 in this proof that $R^F > R^S$. Then consider increasing v_{2L}, which implies that region C is entered. From the proof of Step 2 we know that the increase in v_{2L} increases R^F, and it has no effect on R^S. Hence $R^F > R^S$.

12 Proof of Proposition 6

12.1 Proof of Proposition 6(i)

Consider type 1_j, for $j = L, M, H$. Given that each type of bidder 2 bids v_{1H}, for type 1_j there is no incentive to make a bid different from the own valuation v_{1j}, given that $v_{1j} \leq v_{1H}$.

Now consider type 2_j, for $j = L, M, H$, and notice that bidding $b = v_{1H}$ yields him payoff $v_{2j} - v_{1H} > 0$, whereas $u_{2j}(b) = 0$ if $b < v_{1L}$, $u_{2j}(b) = \lambda_L(v_{2j} - b)$ if $b \in [v_{1L}, v_{1M})$, and $u_{2j}(b) = (\lambda_L + \lambda_M)(v_{2j} - b)$ if $b \in [v_{1M}, v_{1H})$. Given that $\lambda_H v_{2L} + (\lambda_L + \lambda_M)v_{1M} \geq v_{1H}$ and $(\lambda_M + \lambda_H)v_{2L} + \lambda_L v_{1L} \geq v_{1H}$ we infer that $u_{2j}(b) \leq v_{2j} - v_{1H}$ for any $b < v_{1H}$.

12.2 Proof of Proposition 6(ii)

We use $v_L, v_M, v_H + \alpha$ to denote the valuations of bidder 1, and v_L, v_M, v_H to denote the valuations of bidder 2. In Steps 1-3 in this proof we consider the case of a small $\alpha > 0$.

First we show that there exists a BNE in the FPA characterized by three bids b_1, b_2, b_3 such that (i) $v_L < b_1 < b_2 < b_3$; (ii) 1_L bids v_L, 1_M and 1_H play mixed strategies with support $(v_L, b_2]$ for 1_M and $[b_2, b_3]$ for 1_H; (iii) 2_L bids v_L, 2_M and 2_H play mixed strategies with support $[v_L, b_1]$ for 2_M and $[b_1, b_3]$ for 2_H. Then we prove that $\frac{dR^F}{d\alpha}\Big|_{\alpha=0} < 0$ for this BNE, and thus R^F is smaller for a small $\alpha > 0$ than in the case of $\alpha = 0$.

12.2.1 Step 1: Characterization of the equilibrium mixed strategies

Given the supports for the mixed strategies described above, we obtain the following indifference conditions for types 1_M, 1_H, 2_M, 2_H. We use G_{ij} to denote the c.d.f. of the mixed strategy of type i_j, for $i = 1, 2$ and $j = L, M, H$.

Type 1_M:

$$(v_M - b)[\lambda_L + \lambda_M G_{2M}(b)] = (\lambda_L + \lambda_M)(v_M - b_1) \quad \text{for any} \quad b \in (v_L, b_1] \tag{28}$$
$$(v_M - b)[\lambda_L + \lambda_M + \lambda_H G_{2H}(b)] = (\lambda_L + \lambda_M)(v_M - b_1) \quad \text{for any} \quad b \in [b_1, b_2] \tag{29}$$

Type 1_H:

$$(v_H + \alpha - b)[\lambda_L + \lambda_M + \lambda_H G_{2H}(b)] = v_H + \alpha - b_3 \quad \text{for any} \quad b \in [b_2, b_3] \tag{30}$$

Type 2_M:

$$(v_M - b)[\lambda_L + \lambda_M G_{1M}(b)] = \lambda_L(v_M - v_L) \quad \text{for any} \quad b \in [v_L, b_1] \tag{31}$$

Type 2_H:

$$(v_H - b)[\lambda_L + \lambda_M G_{1M}(b)] = v_H - b_3 \quad \text{for any} \quad b \in [b_1, b_2] \tag{32}$$
$$(v_H - b)[\lambda_L + \lambda_M + \lambda_H G_{1H}(b)] = v_H - b_3 \quad \text{for any} \quad b \in [b_2, b_3] \tag{33}$$

Equilibrium rules out mass points at any $b > v_L$, thus each c.d.f. needs to be continuous at bids larger than v_L, and using (31)-(32) we find that G_{1M} is continuous at $b = b_1$ if and only if $\frac{\lambda_L(v_M - v_L)}{v_M - b_1} = \frac{v_H - b_3}{v_H - b_1}$, or

$$\lambda_L(v_H - b_1)(v_M - v_L) = (v_M - b_1)(v_H - b_3) \tag{34}$$

Likewise, (29)-(30) reveal that G_{2H} is continuous at $b = b_2$ if and only if $\frac{(\lambda_L+\lambda_M)(v_M-b_1)}{v_M-b_2} = \frac{v_H+\alpha-b_3}{v_H+\alpha-b_2}$, or

$$(\lambda_L + \lambda_M)(v_M - b_1)(v_H + \alpha - b_2) = (v_H + \alpha - b_3)(v_M - b_2) \tag{35}$$

Finally, $G_{1H}(b_2)$ needs to be 0, and then (33) yields

$$b_3 = \lambda_H v_H + (\lambda_L + \lambda_M)\, b_2 \tag{36}$$

Inserting (36) into (34) and (35) we obtain two equations in the unknowns b_1, b_2:

$$\lambda_L(v_H - b_1)(v_M - v_L) - (\lambda_L + \lambda_M)\,(v_M - b_1)(v_H - b_2) \;=\; 0 \tag{37}$$

$$(\lambda_L + \lambda_M)(v_M - b_1)(v_H + \alpha - b_2) - ((\lambda_L + \lambda_M)(v_H - b_2) + \alpha)(v_M - b_2) \;=\; 0 \tag{38}$$

The system of equations (36)-(38) characterizes the equilibrium values of b_1, b_2, b_3. In the next step we prove that $v_L < b_1 < b_2 < b_3$ for a small $\alpha > 0$, and here we show that these inequalities imply that no incentive to deviate exists for any type, that is the strategies we have described constitute a BNE.

First we notice that the range of bids submitted by bidder 1 and by bidder 2 is $[v_L, b_3]$, thus for no type it is profitable to deviate with a bid below v_L or above b_3. Second, it is useful to take into account the following fact (the proof is immediate after differentiating u with respect to b):

For given $\alpha_1 > 0$, $\alpha_2 > 0$, the function $u(b) = \frac{\alpha_1 - b}{\alpha_2 - b}$, defined for $b \in [0, \alpha_2)$, is increasing if $\alpha_1 > \alpha_2$, is decreasing if $\alpha_1 < \alpha_2$. (39)

Type 1_L. Type 1_L bids v_L with probability one, which gives him payoff zero. Since $u_{1L}(b) < 0$ if he bids $b \in (v_L, b_3]$, he has no incentive to bid in $(v_L, b_3]$.

Type 1_M. Type 1_M plays a mixed strategy with support $(v_L, b_2]$ and his payoff is $(\lambda_L + \lambda_M)(v_M - b_1)$. If instead he bids $b \in (b_2, b_3]$, then $u_{1M}(b) = (v_H + \alpha - b_3)\frac{v_M - b}{v_H + \alpha - b}$ [in view of (30)], which is decreasing in b since $v_M < v_H + \alpha$. This gives type 1_M no incentive to bid in $(b_2, b_3]$. Regarding $b = v_L$, notice that $G_{2M}(v_L) > 0$ since, as we prove in Step 2, $b_1 < v_L + \frac{\lambda_M \Delta}{\lambda_L + \lambda_M}$. Therefore bidding $b = v_L$ implies for type 1_M a positive probability of tying with type 2_M (with a probability of winning in this case equal to $\frac{1}{2}$) and therefore a discrete reduction in the probability of winning with respect to bids slightly above v_L. This makes bidding v_L an unprofitable deviation for 1_M.

Type 1_H. Type 1_H plays a mixed strategy with support $[b_2, b_3]$ and his payoff is $v_H + \alpha - b_3$. If instead he bids $b \in (v_L, b_2)$, then $u_{1H}(b) = (\lambda_L + \lambda_M)(v_M - b_1)\frac{v_H + \alpha - b}{v_M - b}$ [in view of (28) and (29)], which is increasing in b since $v_H + \alpha > v_M$. Therefore type 1_H has no incentive to bid in (v_L, b_2). The same argument described for type 1_M reveal that the bid $b = v_L$ is an unprofitable deviation for 1_H.

Type 2_L. Type 2_L bids v_L with probability one, which gives him payoff zero. Since $u_{2L}(b) < 0$ if he bids $b \in (v_L, b_3]$, he has no incentive to bid in $(v_L, b_3]$.
Type 2_M. Type 2_M plays a mixed strategy with support $[v_L, b_1]$ and his payoff is $\lambda_L(v_M - v_L)$. If instead he bids $b \in (b_1, b_3]$, then $u_{2M}(b) = (v_H - b_3)\frac{v_M - b}{v_H - b}$ [in view of (32)-(33)], which is decreasing in b since $v_M < v_H$. This gives type 2_M no incentive to bid in $(b_1, b_3]$.
Type 2_H. Type 2_H plays a mixed strategy with support $[b_1, b_3]$ and his payoff is $v_H - b_3$. If instead he bids $b \in [v_L, b_1)$, then $u_{2H}(b) = \lambda_L(v_M - v_L)\frac{v_H - b}{v_M - b}$ [in view of (31)], which is increasing in b since $v_H > v_L$. This gives type 2_H no incentive to bid in $[v_L, b_1)$.

12.2.2 Step 2: For a small $\alpha > 0$, the inequalities $v_L < b_1 < b_2 < b_3$ hold

In the following we use $\Delta \equiv v_M - v_L > 0$ and $t \equiv \frac{1}{\Delta}(v_H - v_M) > 0$. The values of b_1, b_2, b_3 depend on α, and therefore we write $b_1(\alpha), b_2(\alpha), b_3(\alpha)$. When $\alpha = 0$ we obtain the symmetric setting, with $b_1(0) = b_2(0) = v_L + \frac{\lambda_M \Delta}{\lambda_L + \lambda_M}$, $b_3(0) = v_L + (\lambda_M + \lambda_H + \lambda_H t)\Delta$. We investigate how b_1, b_2, b_3 depend on α, for a small $\alpha > 0$, by applying the implicit function theorem to (37)-(38) at $\alpha = 0$, $b_1 = b_2 = v_L + \frac{\lambda_M \Delta}{\lambda_L + \lambda_M}$; in this way we obtain $b_1'(0), b_2'(0), b_3'(0)$. To this purpose we denote the left hand sides of (37), (38) with $f_1(b_1, b_2, \alpha), f_2(b_1, b_2, \alpha)$, respectively. Then we obtain

$$
\begin{aligned}
\frac{\partial f_1}{\partial b_1} &= (\lambda_L + \lambda_M)(v_H - b_2) - \lambda_L \Delta, \qquad \frac{\partial f_1}{\partial b_2} = (\lambda_L + \lambda_M)(v_M - b_1), \qquad \frac{\partial f_1}{\partial \alpha} = 0 \\
\frac{\partial f_2}{\partial b_1} &= -(\lambda_L + \lambda_M)(v_H + \alpha - b_2), \qquad \frac{\partial f_2}{\partial b_2} = (\lambda_L + \lambda_M)(v_H + b_1 - 2b_2) + \alpha, \\
\frac{\partial f_2}{\partial \alpha} &= b_2 - (\lambda_L + \lambda_M) b_1 - \lambda_H v_M
\end{aligned}
$$

We evaluate these derivatives at $\alpha = 0$, $b_1 = b_2 = v_L + \frac{\lambda_M \Delta}{\lambda_L + \lambda_M}$ and find

$$
\begin{aligned}
\begin{bmatrix} b_1'(0) \\ b_2'(0) \end{bmatrix} &= -\begin{bmatrix} (\lambda_L + \lambda_M)\Delta t & \lambda_L \Delta \\ -(\lambda_L + t\lambda_L + t\lambda_M)\Delta & (\lambda_L + t\lambda_L + t\lambda_M)\Delta \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\frac{\lambda_H \lambda_L}{\lambda_L + \lambda_M}\Delta \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\lambda_H \lambda_L^2}{(\lambda_L + t\lambda_L + t\lambda_M)^2(\lambda_L + \lambda_M)} \\ \frac{\lambda_H \lambda_L t}{(\lambda_L + t\lambda_L + t\lambda_M)^2} \end{bmatrix}
\end{aligned}
$$

Using (36) we see that $b_3'(0) = \frac{(\lambda_L + \lambda_M)\lambda_H \lambda_L t}{(\lambda_L + t\lambda_L + t\lambda_M)^2}$. In next step we use $b_1'(0), b_2'(0), b_3'(0)$ to derive $\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0}$.

12.2.3 Step 3: Proof that $\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0} < 0$

We define $G_i(b)$ as $\lambda_L G_{iL}(b) + \lambda_M G_{iM}(b) + \lambda_H G_{iH}(b)$ for $i = 1, 2$, so that $G(b) \equiv G_1(b) G_2(b)$ is the c.d.f. of the winning bid. In particular, from (28)-(33) we obtain

$$
G(b) = \begin{cases}
[\lambda_L + \lambda_M G_{1M}(b)][\lambda_L + \lambda_M G_{2M}(b)] = \lambda_L(\lambda_L + \lambda_M)\frac{\Delta[v_M - b_1(\alpha)]}{(v_M - b)^2} & \text{if } b \in [v_L, b_1(\alpha)) \\
[\lambda_L + \lambda_M G_{1M}(b)][\lambda_L + \lambda_M + \lambda_H G_{2H}(b)] = (\lambda_L + \lambda_M)\frac{[v_H - b_3(\alpha)][v_M - b_1(\alpha)]}{(v_M - b)(v_H - b)} & \text{if } b \in [b_1(\alpha), b_2(\alpha)) \\
[\lambda_L + \lambda_M + \lambda_H G_{1H}(b)][\lambda_L + \lambda_M + \lambda_H G_{2H}(b)] = \frac{[v_H - b_3(\alpha)][v_H + \alpha - b_3(\alpha)]}{(v_H - b)(v_H + \alpha - b)} & \text{if } b \in [b_2(\alpha), b_3(\alpha)]
\end{cases}
$$

Since

$$\begin{aligned}
R^F &= v_L G(v_L) + \int_{v_L}^{b_3(\alpha)} b dG(b) = b_3(\alpha) - \int_{v_L}^{b_3(\alpha)} G(b) db \\
&= b_3(\alpha) - \int_{v_L}^{b_1(\alpha)} \lambda_L(\lambda_L+\lambda_M)\frac{\Delta[v_M - b_1(\alpha)]}{(v_M-b)^2} db - \int_{b_1(\alpha)}^{b_2(\alpha)} (\lambda_L+\lambda_M)\frac{[v_H-b_3(\alpha)][v_M-b_1(\alpha)]}{(v_M-b)(v_H-b)} db \\
&\quad - \int_{b_2(\alpha)}^{b_3(\alpha)} \frac{[v_H-b_3(\alpha)][v_H+\alpha-b_3(\alpha)]}{(v_H-b)(v_H+\alpha-b)} db
\end{aligned}$$

we can use this expression to evaluate $\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0}$.

- The derivative of $\int_{v_L}^{b_1(\alpha)} \lambda_L(\lambda_L+\lambda_M)\frac{\Delta[v_M-b_1(\alpha)]}{(v_M-b)^2} db$ with respect to α is $\lambda_L(\lambda_L+\lambda_M)\Delta[\frac{b_1'(\alpha)}{v_M-b_1} - \int_{v_L}^{b_1(\alpha)} \frac{b_1'(\alpha)}{(v_M-b)^2} db]$ and at $\alpha = 0$ it boils down to $-\frac{\lambda_H\lambda_L^3}{(\lambda_L+t\lambda_L+t\lambda_M)^2}$.

- The derivative of $\int_{b_1(\alpha)}^{b_2(\alpha)} (\lambda_L+\lambda_M)\frac{[v_H-b_3(\alpha)][v_M-b_1(\alpha)]}{(v_M-b)(v_H-b)} db$ with respect to α is

$$\begin{aligned}
(\lambda_L+\lambda_M)\{&\frac{[v_H-b_3(\alpha)][v_M-b_1(\alpha)]}{[v_M-b_2(\alpha)][v_H-b_2(\alpha)]} b_2'(\alpha) - \frac{v_H-b_3(\alpha)}{v_H-b_1(\alpha)} b_1'(\alpha) \\
&- \int_{b_1(\alpha)}^{b_2(\alpha)} \frac{b_3'(\alpha)[v_M-b_1(\alpha)] + b_1'(\alpha)[v_H-b_3(\alpha)]}{(v_M-b)(v_H-b)} db\}
\end{aligned}$$

  and at $\alpha = 0$ it boils down to $\frac{(\lambda_L+\lambda_M)\lambda_L\lambda_H}{\lambda_L+t\lambda_L+t\lambda_M}$.

- The derivative of $\int_{b_2(\alpha)}^{b_3(\alpha)} \frac{[v_H-b_3(\alpha)][v_H+\alpha-b_3(\alpha)]}{(v_H-b)(v_H+\alpha-b)} db$ with respect to α is

$$\begin{aligned}
&b_3'(\alpha) - \frac{[v_H-b_3(\alpha)][v_H+\alpha-b_3(\alpha)]b_2'(\alpha)}{[v_H-b_2(\alpha)][v_H+\alpha-b_2(\alpha)]} \\
&+ \int_{b_2(\alpha)}^{b_3(\alpha)} \frac{\{[v_H-b_3(\alpha)][1-2b_3'(\alpha)] - \alpha b_3'(\alpha)\}(v_H+\alpha-b) - [v_H-b_3(\alpha)][v_H+\alpha-b_3(\alpha)]}{(v_H-b)(v_H+\alpha-b)^2} db
\end{aligned}$$

  and at $\alpha = 0$ it boils down to $\frac{\lambda_H^2\lambda_L(\lambda_L+\lambda_M)t}{(\lambda_L+t\lambda_L+t\lambda_M)^2} + \frac{\lambda_H^2(t\lambda_L+t\lambda_M-\lambda_L)^2}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}$ which is equal to $\lambda_H^2\frac{(\lambda_L+\lambda_M)^2t^2+\lambda_L^2}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}$.

Therefore

$$\begin{aligned}
\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0} &= \frac{(\lambda_L+\lambda_M)\lambda_H\lambda_L t}{(\lambda_L+t\lambda_L+t\lambda_M)^2} + \frac{\lambda_H\lambda_L^3}{(\lambda_L+t\lambda_L+t\lambda_M)^2} - \frac{(\lambda_L+\lambda_M)\lambda_L\lambda_H}{\lambda_L+t\lambda_L+t\lambda_M} - \lambda_H^2\frac{(\lambda_L+\lambda_M)^2 t^2 + \lambda_L^2}{2(\lambda_L+t\lambda_L+t\lambda_M)^2} \\
&= -\lambda_H \frac{\lambda_H(\lambda_L+\lambda_M)^2 (t - \frac{\lambda_L}{\lambda_L+\lambda_M})^2 + 2\lambda_L^2\lambda_M}{2(\lambda_L+t\lambda_L+t\lambda_M)^2} < 0
\end{aligned}$$

On the other hand, R^S does not change if α increases from 0 to a positive value, thus $R^S > R^F$ for a small $\alpha > 0$.

12.2.4 Step 4: The case of a small reduction in v_{1H}

Consider the symmetric setting such that $v_{1L} = v_{2L} = v_L$, $v_{1M} = v_{2M} = v_M$, $v_{1H} = v_{2H} = v_H$; then $R^F = v_L + (\lambda_M + \lambda_H)^2\Delta + \lambda_H^2\Delta t$. We need to prove that R^F is larger in this case than if v_{1H} is reduced to $v_H - \alpha$, for a small $\alpha > 0$. In order to prove the latter property, consider first the symmetric setting in which $v_{1L} = v_{2L} = v_L$, $v_{1M} = v_{2M} = v_M$, $v_{1H} = v_{2H} = v_H - \alpha$; then $R^F = v_L + (\lambda_M + \lambda_H)^2\Delta + \lambda_H^2(\Delta t - \alpha)$. Now increase v_{2H} from $v_H - \alpha$ to v_H. By Steps 1-3 in this proof, the effect is that R^F is reduced below $v_L + (\lambda_M + \lambda_H)^2\Delta + \lambda_H^2(\Delta t - \alpha)$, which guarantees that R^F is smaller than $v_L + (\lambda_M + \lambda_H)^2\Delta + \lambda_H^2\Delta t$.

12.3 Proof of Proposition 6(iii)

In this proof we use v_L, v_M, v_H to denote the valuations of bidder 1, and $v_L + y\alpha, v_M, v_H - \alpha$ to denote the valuations of bidder 2, for an arbitrary $y > 0$ and a small $\alpha > 0$.

First we show that there exists a BNE in the FPA characterized by three bids b_1, b_2, b_3 such that (i) $v_L < b_1 < b_2 < b_3$; (ii) 1_L bids v_L, 1_M and 1_H play mixed strategies with support $(v_L, b_2]$ for 1_M and $[b_2, b_3]$ for 1_H; (iii) 2_L bids v_L, 2_M and 2_H play mixed strategies with supports $[v_L, b_1]$ for 2_M and $[b_1, b_3]$ for 2_H. Then we evaluate $\frac{dR^F}{d\alpha}\Big|_{\alpha=0}$ for this BNE and prove that $\frac{dR^F}{d\alpha}\Big|_{\alpha=0} < \frac{dRS}{d\alpha}\Big|_{\alpha=0}$. Thus $R^F < R^S$ for a small $\alpha > 0$.

12.3.1 Step 1: Characterization of the equilibrium mixed strategies

Given the supports for the mixed strategies described above, we obtain the following indifference conditions for types 1_M, 1_H, 2_M, 2_H. We use G_{ij} to denote the c.d.f. of the mixed strategy of type i_j, for $i = 1, 2$ and $j = L, M, H$.

Type 1_M:

$$(v_M - b)[\lambda_L + \lambda_M G_{2M}(b)] = (\lambda_L + \lambda_M)(v_M - b_1) \quad \text{for any} \quad b \in [v_L, b_1] \tag{40}$$
$$(v_M - b)[\lambda_L + \lambda_M + \lambda_H G_{2H}(b)] = (\lambda_L + \lambda_M)(v_M - b_1) \quad \text{for any} \quad b \in [b_1, b_2] \tag{41}$$

Type 1_H:

$$(v_H - b)[\lambda_L + \lambda_M + \lambda_H G_{2H}(b)] = v_H - b_3 \quad \text{for any} \quad b \in [b_2, b_3] \tag{42}$$

Type 2_M:

$$(v_M - b)[\lambda_L + \lambda_M G_{1M}(b)] = \lambda_L(v_M - v_L) \quad \text{for any} \quad b \in [v_L, b_1] \tag{43}$$

Type 2_H:

$$(v_H - \alpha - b)[\lambda_L + \lambda_M G_{1M}(b)] = v_H - \alpha - b_3 \quad \text{for any} \quad b \in [b_1, b_2] \tag{44}$$
$$(v_H - \alpha - b)[\lambda_L + \lambda_M + \lambda_H G_{1H}(b)] = v_H - \alpha - b_3 \quad \text{for any} \quad b \in [b_2, b_3] \tag{45}$$

Equilibrium rules out mass points at any $b > v_L$, thus each c.d.f. needs to be continuous at bids larger than v_L, and using (43)-(44) we find that G_{1M} is continuous at $b = b_1$ if and only if $\frac{\lambda_L(v_M - v_L)}{v_M - b_1} = \frac{v_H - \alpha - b_3}{v_H - \alpha - b_1}$, or

$$\lambda_L(v_H - \alpha - b_1)(v_M - v_L) - (v_M - b_1)(v_H - \alpha - b_3) = 0 \tag{46}$$

Likewise, (41)-(42) reveal that G_{2H} is continuous at $b = b_2$ if and only if $\frac{(\lambda_L + \lambda_M)(v_M - b_1)}{v_M - b_2} = \frac{v_H - b_3}{v_H - b_2}$, or

$$(\lambda_L + \lambda_M)(v_M - b_1)(v_H - b_2) - (v_H - b_3)(v_M - b_2) \tag{47}$$

Finally, $G_{1H}(b_2)$ needs to be 0, and then (45) yields

$$b_3 = \lambda_H(v_H - \alpha) + (\lambda_L + \lambda_M)\, b_2 \tag{48}$$

Inserting (48) into (46) and (47) we obtain two equations in the unknowns b_1, b_2:

$$\lambda_L(v_H - \alpha - b_1)(v_M - v_L) - (\lambda_L + \lambda_M)\,(v_M - b_1)(v_H - \alpha - b_2) = 0 \tag{49}$$

$$(\lambda_L + \lambda_M)(v_M - b_1)(v_H - b_2) - ((1 - \lambda_H)(v_H - b_2) + \lambda_H \alpha)(v_M - b_2) = 0 \tag{50}$$

The system of equations (48)-(50) characterizes the equilibrium values of b_1, b_2, b_3. It is important to notice that the valuation of type 2_L, $v_L + y\alpha$, plays no role. In the next step we prove that $v_L < b_1 < b_2 < b_3$ for a small $\alpha > 0$, and here we show that these inequalities imply that no incentive to deviate exists for any type, that is the strategies we have described constitute a BNE.

First we notice that the range of bids submitted by bidder 1 and by bidder 2 is $[v_L, b_3]$, thus for no type it is profitable to deviate with a bid below v_L or above b_3. Second, it is useful to take into account fact (39).

Type 1_L. Type 1_L bids v_L with probability one, which gives him payoff zero. Since $u_{1L}(b) < 0$ if he bids $b \in (v_L, b_3]$, he has no incentive to bid in $(v_L, b_3]$.

Type 1_M. Type 1_M plays a mixed strategy with support $(v_L, b_2]$ and his payoff is $(\lambda_L + \lambda_M)(v_M - b_1)$. If instead he bids $b \in (b_2, b_3]$, then $u_{1M}(b) = (v_H - b_3)\frac{v_M - b}{v_H - b}$ [in view of (42)], which is decreasing in b since $v_M < v_H$. This gives type 1_M no incentive to bid in $(b_2, b_3]$. Regarding $b = v_L$, notice that $G_{2M}(v_L) > 0$ since, as we prove in Step 2, $b_1 < v_L + \frac{\lambda_M \Delta}{\lambda_L + \lambda_M}$. Therefore bidding $b = v_L$ implies for type 1_M a positive probability of tying with type 2_M (with a probability of winning in this case equal to $\frac{1}{2}$) and therefore a discrete reduction in the probability of winning with respect to bids slightly above v_L. This makes bidding v_L an unprofitable deviation for 1_M.

Type 1_H. Type 1_H plays a mixed strategy with support $[b_2, b_3]$ and his payoff is $v_H - b_3$. If instead he bids $b \in (v_L, b_2)$, then $u_{1H}(b) = (\lambda_L + \lambda_M)(v_M - b_1)\frac{v_H - b}{v_M - b}$ [in view of (40) and (41)], which is increasing in b since $v_H > v_M$. Therefore type 1_H has no incentive to bid in (v_L, b_2). The same argument described for type 1_M reveal that the bid $b = v_L$ is an unprofitable deviation for 1_H.

Type 2_L. Type 2_L bids v_L with probability one, which gives him payoff $\lambda_L y\alpha$. If instead he bids $b \in (v_L, v_L + y\alpha]$, then $u_{2L}(b) = \lambda_L(v_M - v_L)\frac{v_L + y\alpha - b}{v_M - b}$ [in view of (43)], which is decreasing in b

since $v_L + y\alpha < v_M$. Hence 2_L has no incentive to bid in $(v_L, v_L + y\alpha]$, and $u_{2L}(b) < 0$ if he bids $b \in (v_L + y\alpha, b_3]$.

Type 2_M. Type 2_M plays a mixed strategy with support $[v_L, b_1]$ and his payoff is $\lambda_L(v_M - v_L)$. If instead he bids $b \in (b_1, b_3]$, then $u_{2M}(b) = (v_H - \alpha - b_3)\frac{v_M - b}{v_H - \alpha - b}$ [in view of (44)-(45)], which is decreasing in b since $v_M < v_H - \alpha$. This gives type 2_M no incentive to bid in $(b_1, b_3]$.

Type 2_H. Type 2_H plays a mixed strategy with support $[b_1, b_3]$ and his payoff is $v_H - \alpha - b_3$. If instead he bids $b \in [v_L, b_1)$, then $u_{2H}(b) = \lambda_L(v_M - v_L)\frac{v_H - \alpha - b}{v_M - b}$ [in view of (43)], which is increasing in b since $v_H > v_M$. This gives type 2_H no incentive to bid in $[v_L, b_1)$.

12.3.2 Step 2: For a small $\alpha > 0$, we have $v_L < b_1 < b_2 < b_3$

In the following we use $\Delta \equiv v_M - v_L > 0$ and $t \equiv \frac{1}{\Delta}(v_H - v_M) > 0$. The values of b_1, b_2, b_3 depend on α, and therefore we write $b_1(\alpha), b_2(\alpha), b_3(\alpha)$. When $\alpha = 0$ we obtain the symmetric setting, with $b_1(0) = b_2(0) = v_L + \frac{\lambda_M \Delta}{\lambda_L + \lambda_M}$, $b_3(0) = v_L + (\lambda_M + \lambda_H + t\lambda_H)\Delta$. We investigate how b_1, b_2, b_3 depend on α, for a small $\alpha > 0$, by applying the implicit function theorem to (49)-(50) at $\alpha = 0$, $b_1 = b_2 = v_L + \frac{\lambda_M \Delta}{\lambda_L + \lambda_M}$; in this way we obtain $b_1'(0), b_2'(0), b_3'(0)$. To this purpose we denote the left hand sides of (49),(50) with $f_1(b_1, b_2, \alpha), f_2(b_1, b_2, \alpha)$, respectively. Then we obtain

$$\begin{aligned}
\frac{\partial f_1}{\partial b_1} &= (1-\lambda_H)(v_H - \alpha - b_2) - \lambda_L \Delta, \quad \frac{\partial f_1}{\partial b_2} = (1-\lambda_H)(v_M - b_1), \quad \frac{\partial f_1}{\partial \alpha} = (1-\lambda_H)(v_M - b_1) - \lambda_L \Delta \\
\frac{\partial f_2}{\partial b_1} &= -(1-\lambda_H)(v_H - b_2), \quad \frac{\partial f_2}{\partial b_2} = (1-\lambda_H)(v_H + b_1 - 2b_2) + \lambda_H \alpha, \quad \frac{\partial f_2}{\partial \alpha} = -\lambda_H(v_M - b_2)
\end{aligned}$$

We evaluate these derivatives at $\alpha = 0$, $b_1 = b_2 = v_L + \frac{\lambda_M \Delta}{\lambda_L + \lambda_M}$ and find

$$\begin{aligned}
\begin{bmatrix} b_1'(0) \\ b_2'(0) \end{bmatrix} &= -\begin{bmatrix} (\lambda_L + \lambda_M)\Delta t & \lambda_L \Delta \\ -(\lambda_L + t\lambda_L + t\lambda_M)\Delta & (\lambda_L + t\lambda_L + t\lambda_M)\Delta \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\frac{\lambda_H \lambda_L}{\lambda_L + \lambda_M}\Delta \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\lambda_H \lambda_L^2}{(\lambda_L + t\lambda_L + t\lambda_M)^2(\lambda_L + \lambda_M)} \\ \frac{\lambda_H \lambda_L t}{(\lambda_L + t\lambda_L + t\lambda_M)^2} \end{bmatrix}
\end{aligned}$$

Using (48) we see that $b_3'(0) = -\lambda_H + \frac{(\lambda_L + \lambda_M)\lambda_H \lambda_L t}{(\lambda_L + t\lambda_L + t\lambda_M)^2} = -\lambda_H \frac{(\lambda_L + \lambda_M)^2 t^2 + \lambda_L(\lambda_L + \lambda_M)t + \lambda_L^2}{(\lambda_L + t\lambda_L + t\lambda_M)^2}$. In next step we use $b_1'(0), b_2'(0), b_3'(0)$ to derive $\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0}$.

12.3.3 Step 3: Evaluation of $\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0}$

We define $G_i(b)$ as $\lambda_L G_{iL}(b) + \lambda_M G_{iM}(b) + \lambda_H G_{iH}(b)$ for $i = 1, 2$, so that $G(b) \equiv G_1(b)G_2(b)$ is the c.d.f. of the winning bid. In particular, from (40)-(45) we obtain

$$G(b) = \begin{cases} [\lambda_L + \lambda_M G_{1M}(b)][\lambda_L + \lambda_M G_{2M}(b)] = \lambda_L(\lambda_L + \lambda_M)\frac{v_M - v_L}{v_M - b}\frac{v_M - b_1(\alpha)}{v_M - b} & \text{if } b \in [v_L, b_1(\alpha)) \\ [\lambda_L + \lambda_M G_{1M}(b)][\lambda_L + \lambda_M + \lambda_H G_{2H}(b)] = (\lambda_L + \lambda_M)\frac{v_H - \alpha - b_3(\alpha)}{v_H - \alpha - b}\frac{v_M - b_1(\alpha)}{v_M - b} & \text{if } b \in [b_1(\alpha), b_2(\alpha)) \\ [\lambda_L + \lambda_M + \lambda_H G_{1H}(b)][\lambda_L + \lambda_M + \lambda_H G_{2H}(b)] = \frac{v_H - \alpha - b_3(\alpha)}{v_H - \alpha - b}\frac{v_H - b_3(\alpha)}{v_H - b} & \text{if } b \in [b_2(\alpha), b_3(\alpha)] \end{cases}$$

Since

$$\begin{aligned}
R^F &= v_L G(v_L) + \int_{v_L}^{b_3(\alpha)} b dG(b) = b_3(\alpha) - \int_{v_L}^{b_3(\alpha)} G(b) db \\
&= b_3(\alpha) - \int_{v_L}^{b_1(\alpha)} \lambda_L(\lambda_L+\lambda_M) \frac{\Delta}{v_M - b} \frac{v_M - b_1(\alpha)}{v_M - b} db \\
&- \int_{b_1(\alpha)}^{b_2(\alpha)} (\lambda_L+\lambda_M) \frac{v_H - \alpha - b_3(\alpha)}{v_H - \alpha - b} \frac{v_M - b_1(\alpha)}{v_M - b} db - \int_{b_2(\alpha)}^{b_3(\alpha)} \frac{v_H - \alpha - b_3(\alpha)}{v_H - \alpha - b} \frac{v_H - b_3(\alpha)}{v_H - b} db
\end{aligned}$$

- The derivative of $\int_{v_L}^{b_1(\alpha)} \frac{\lambda_L \Delta}{v_M - b}(\lambda_L+\lambda_M)\frac{v_M - b_1(\alpha)}{v_M - b} db$ with respect to α is

$$\lambda_L(\lambda_L+\lambda_M) b_1'(\alpha)\Delta[\frac{1}{v_M - b_1} - \int_{v_L}^{b_1(\alpha)} \frac{1}{(v_M - b)^2} db]$$

  and at $\alpha = 0$ it boils down to $-\frac{\lambda_H \lambda_L^3}{(\lambda_L + t\lambda_L + t\lambda_M)^2}$.

- The derivative of $\int_{b_1(\alpha)}^{b_2(\alpha)} \frac{v_H - \alpha - b_3(\alpha)}{v_H - \alpha - b}(\lambda_L+\lambda_M)\frac{v_M - b_1(\alpha)}{v_M - b} db$ with respect to α is

$$(\lambda_L+\lambda_M)\left(\begin{array}{c} \frac{[v_H-\alpha-b_3(\alpha)][v_M-b_1(\alpha)]}{[v_M-b_2(\alpha)][v_H-\alpha-b_2(\alpha)]} b_2'(\alpha) - \frac{v_H-\alpha-b_3(\alpha)}{v_H-\alpha-b_1(\alpha)} b_1'(\alpha) \\ - \int_{b_1(\alpha)}^{b_2(\alpha)} \frac{\{-[1+b_3'(\alpha)][v_M-b_1(\alpha)]-b_1'(\alpha)[v_H-\alpha-b_3(\alpha)]\}(v_H-\alpha-b)+[v_H-\alpha-b_3(\alpha)][v_M-b_1(\alpha)]}{(v_M-b)(v_H-\alpha-b)^2} db \end{array}\right)$$

  and at $\alpha = 0$ it boils down to $\frac{(\lambda_L+\lambda_M)\lambda_L\lambda_H}{\lambda_L + t\lambda_L + t\lambda_M}$.

- The derivative of $\int_{b_2(\alpha)}^{b_3(\alpha)} \frac{v_H - \alpha - b_3(\alpha)}{v_H - \alpha - b}\frac{v_H - b_3(\alpha)}{v_H - b} db$ with respect to α is

$$b_3'(\alpha) - \frac{[v_H - \alpha - b_3(\alpha)][v_H - b_3(\alpha)]}{[v_H - \alpha - b_2(\alpha)][v_H - b_2(\alpha)]} b_2'(\alpha)$$

$$+ \int_{b_2(\alpha)}^{b_3(\alpha)} \frac{\{\alpha b_3'(\alpha) + [b_3(\alpha) - v_H][1 + 2b_3'(\alpha)]\}(v_H - \alpha - b) + [v_H - \alpha - b_3(\alpha)][v_H - b_3(\alpha)]}{(v_H - b)(v_H - \alpha - b)^2} db$$

  and at $\alpha = 0$ it boils down to $-\lambda_H \frac{(\lambda_L+\lambda_M)^2 t^2 + \lambda_L(2-\lambda_H)(\lambda_L+\lambda_M)t + \lambda_L^2}{(\lambda_L+t\lambda_L+t\lambda_M)^2} + \lambda_H^2 \frac{3(\lambda_L+\lambda_M)^2t^2 + 2\lambda_L(\lambda_L+\lambda_M)t + 3\lambda_L^2}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}$, which is equal to $\lambda_H \frac{(3\lambda_H - 2)(\lambda_L^2 + (\lambda_L+\lambda_M)^2t^2) - 4\lambda_L(\lambda_L+\lambda_M)^2 t}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}$.

Therefore

$$\begin{aligned}
\frac{dR^F}{d\alpha} &= -\lambda_H \frac{t^2(1-\lambda_H)^2 + \lambda_L(\lambda_L + t\lambda_L + t\lambda_M)}{(\lambda_L + t\lambda_L + t\lambda_M)^2} + \frac{\lambda_H\lambda_L^3}{(\lambda_L + t\lambda_L + t\lambda_M)^2} - \frac{(1-\lambda_H)\lambda_L\lambda_H}{\lambda_L + t\lambda_L + t\lambda_M} \\
&\quad -\lambda_H \frac{(1 - 3\lambda_L - 3\lambda_M)\left(\lambda_L^2 + (1-\lambda_H)^2 t^2\right) - 4\lambda_L(1-\lambda_H)^2 t}{2\left(\lambda_L + t\lambda_L + t\lambda_M\right)^2} \\
&= -\lambda_H \frac{\lambda_L^2\left(3\lambda_H + 2\lambda_M\right) + \lambda_H(1-\lambda_H)(2\lambda_L + 3(1-\lambda_H)t)t}{2\left(\lambda_L + t\lambda_L + t\lambda_M\right)^2}
\end{aligned}$$

12.3.4 Step 4: $\left.\frac{d(R^S-R^F)}{d\alpha}\right|_{\alpha=0} > 0$

It is straightforward to see that

$$\begin{aligned} R^S &= \lambda_L v_L + \lambda_M\lambda_L(v_L+y\alpha) + \lambda_M(\lambda_M+\lambda_H)v_M + \lambda_H\lambda_L(v_L+y\alpha) + \lambda_H\lambda_M v_M + \lambda_H^2(v_H-\alpha) \\ &= v_L + ((1-\lambda_L)^2 + t\lambda_H^2)\Delta + (y\lambda_L(1-\lambda_L)-\lambda_H^2)\alpha \end{aligned}$$

and thus $\frac{dR^S}{d\alpha} = y\lambda_L(1-\lambda_L)-\lambda_H^2$. The inequality $\left.\frac{d(R^S-R^F)}{d\alpha}\right|_{\alpha=0} > 0$ is equivalent to

$$y\lambda_L(1-\lambda_L)-\lambda_H^2+\lambda_H\frac{\lambda_L^2\left(3\lambda_H+2\lambda_M\right)+\lambda_H(1-\lambda_H)(2\lambda_L+3(1-\lambda_H)t)t}{2\left(\lambda_L+t\lambda_L+t\lambda_M\right)^2} > 0$$

For $y=0$, the left hand side in this inequality is $\lambda_H\frac{\lambda_H(1-\lambda_H)^2(t-\frac{\lambda_L}{1-\lambda_H})^2+2\lambda_L^2\lambda_M}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}$, which is positive and thus $\left.\frac{d(R^S-R^F)}{d\alpha}\right|_{\alpha=0} > 0$ for any $y \geq 0$.

12.4 Proof of Proposition 6(iv)

In this proof we use v_L, v_M, v_H to denote the valuations of bidder 1 and $v_L+\alpha, v_M+\alpha, v_H+\alpha$ to denote the valuations of bidder 2, for a small $\alpha > 0$.

First we show that there exists a BNE in the FPA characterized by four bids b_1, b_2, b_3, b_4 such that (i) $v_L < b_1 < b_2 < b_3 < b_4$; (ii) 1_L bids v_L, 1_M and 1_H play mixed strategies with support $[v_L, b_2]$ for 1_M and $[b_2, b_4]$ for 1_H; (iii) 2_L, 2_M, 2_H play mixed strategies with support $[v_L, b_1]$ for 2_L, $[b_1, b_3]$ for 2_M, $[b_3, b_4]$ for 2_H. Then we evaluate $\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0}$ for this BNE and prove that $\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0} < \left.\frac{dRS}{d\alpha}\right|_{\alpha=0}$. Thus $R^F < R^S$ for a small $\alpha > 0$.

12.4.1 Step 1: Characterization of the equilibrium mixed strategies

Given the supports for the mixed strategies described above, we obtain the following indifference conditions for types 1_M, 1_H, 2_L, 2_M, 2_H. We use G_{ij} to denote the c.d.f. of the mixed strategy of type i_j, for $i = 1, 2$ and $j = L, M, H$.

Type 1_M:

$$(v_M-b)\lambda_L G_{2L}(b) = \lambda_L(v_M-b_1) \quad \text{for any} \quad b \in [v_L, b_1] \tag{51}$$

$$(v_M-b)[\lambda_L+\lambda_M G_{2M}(b)] = \lambda_L(v_M-b_1) \quad \text{for any} \quad b \in (b_1, b_2] \tag{52}$$

Type 1_H:

$$(v_H-b)[\lambda_L+\lambda_M G_{2M}(b)] = v_H-b_4 \quad \text{for any} \quad b \in [b_2, b_3] \tag{53}$$

$$(v_H-b)[\lambda_L+\lambda_M+\lambda_H G_{2H}(b)] = v_H-b_4 \quad \text{for any} \quad b \in (b_3, b_4] \tag{54}$$

Type 2_L:

$$(v_L+\alpha-b)[\lambda_L+\lambda_M G_{1M}(b)] = \lambda_L(v_L+\alpha-v_L) \quad \text{for any} \quad b \in [v_L, b_1] \tag{55}$$

Type 2_M:

$$
\begin{aligned}
(v_M+\alpha-b)[\lambda_L+\lambda_M G_{1M}(b)] &= (\lambda_L+\lambda_M)(v_M+\alpha-b_2) \quad \text{for any} \quad b\in[b_1,b_2] \quad (56)\\
(v_M+\alpha-b)[\lambda_L+\lambda_M+\lambda_H G_{1H}(b)] &= (\lambda_L+\lambda_M)(v_M+\alpha-b_2) \quad \text{for any} \quad b\in(b_2,b_3] \quad (57)
\end{aligned}
$$

Type 2_H:

$$
(v_H+\alpha-b)[\lambda_L+\lambda_M+\lambda_H G_{1H}(b)] = v_H+\alpha-b_4 \quad \text{for any} \quad b\in[b_3,b_4] \tag{58}
$$

Equilibrium rules out mass points at any $b>v_L$, thus each c.d.f. needs to be continuous at bids larger than v_L, and using (55)-(56) we find that G_{1M} is continuous at $b=b_1$ if and only if $\frac{\lambda_L\alpha}{v_L+\alpha-b_1}=\frac{(\lambda_L+\lambda_M)(v_M+\alpha-b_2)}{v_M+\alpha-b_1}$, or

$$
\lambda_L\alpha(v_M+\alpha-b_1)=(\lambda_L+\lambda_M)(v_M+\alpha-b_2)(v_L+\alpha-b_1) \tag{59}
$$

Likewise, (57)-(58) reveal that G_{1H} is continuous at $b=b_3$ if and only if $\frac{(\lambda_L+\lambda_M)(v_M+\alpha-b_2)}{v_M+\alpha-b_3}=\frac{v_H+\alpha-b_4}{v_H+\alpha-b_3}$, or

$$
(\lambda_L+\lambda_M)(v_M+\alpha-b_2)(v_H+\alpha-b_3)=(v_M+\alpha-b_3)(v_H+\alpha-b_4) \tag{60}
$$

Likewise, (52)-(53) reveal that G_{2M} is continuous at $b=b_2$ if and only if $\frac{\lambda_L(v_M-b_1)}{v_M-b_2}=\frac{v_H-b_4}{v_H-b_2}$, or

$$
\lambda_L(v_M-b_1)(v_H-b_2)=(v_H-b_4)(v_M-b_2) \tag{61}
$$

Finally, $G_{2H}(b_3)$ needs to be 0 and then (54) yields

$$
b_4=\lambda_H v_H+(\lambda_L+\lambda_M)\,b_3 \tag{62}
$$

Inserting (62) into (59)-(61) we obtain three equations in the unknowns b_1,b_2,b_3:

$$
\begin{aligned}
\lambda_L\alpha(v_M+\alpha-b_1)-(\lambda_L+\lambda_M)(v_M+\alpha-b_2)(v_L+\alpha-b_1) &= 0 \quad (63)\\
(\lambda_L+\lambda_M)(v_M+\alpha-b_2)(v_H+\alpha-b_3)-(v_M+\alpha-b_3)((1-\lambda_H)(v_H-b_3)+\alpha) &= 0 \quad (64)\\
\lambda_L(v_M-b_1)(v_H-b_2)-(\lambda_L+\lambda_M)\,(v_H-b_3)\,(v_M-b_2) &= 0 \quad (65)
\end{aligned}
$$

The system of equations (62)-(65) characterizes the equilibrium values of b_1,b_2,b_3,b_4. In the next step we prove that $v_L<b_1<b_2<b_3<b_4$ for a small $\alpha>0$, and here we show that these inequalities imply that no incentive to deviate exists for any type, that is the strategies we have described constitute a BNE.

First notice that the range of bids submitted by bidder 1 and by bidder 2 is $[v_L,b_4]$, thus for no type it is profitable to deviate with a bid below v_L or above b_4. Second, it is useful to take into account fact (39).

Type 1_L. Type 1_L bids v_L with probability one, which gives him payoff zero. Since $u_{1L}(b)<0$ if bids $b\in(v_L,b_4]$, he has no incentive to bid in $(v_L,b_4]$.

Type 1_M. Type 1_M plays a mixed strategy with support $[v_L, b_2]$ and his payoff is $\lambda_L(v_M - b_1)$. If instead he bids $b \in (b_2, b_4]$, then $u_{1M}(b) = (v_H - b_4)\frac{v_M - b}{v_H - b}$ [in view of (53) and (54)], which is decreasing in b since $v_M < v_H$. This gives type 1_M no incentive to bid in $(b_2, b_4]$.
Type 1_H. Type 1_H plays a mixed strategy with support $[b_2, b_4]$ and his payoff is $v_H - b_4$. If instead he bids $b \in [v_L, b_2)$, then $u_{1H}(b) = \lambda_L(v_M - b_1)\frac{v_H - b}{v_M - b}$ [in view of (51) and (52)], which is increasing in b since $v_H > v_M$. This gives type 1_H no incentive to bid in $[v_L, b_2)$.
Type 2_L. Type 2_L plays a mixed strategy with support $[v_L, b_1]$ and his payoff is $\lambda_L\alpha$. If instead he bids $b \in (b_1, b_3]$, then $u_{2L}(b) = (\lambda_L + \lambda_M)(v_M + \alpha - b_2)\frac{v_L + \alpha - b}{v_M + \alpha - b}$ [in view of (56) and (57)], which is decreasing in b since $v_L + \alpha < v_M + \alpha$. Moreover, if 2_L bids $b \in (b_3, b_4]$ then $u_{2L}(b) = (v_H + \alpha - b_4)\frac{v_L + \alpha - b}{v_H + \alpha - b}$, which is decreasing in b since $v_L + \alpha < v_H + \alpha$. Therefore type 2_L has no incentive to bid in $(b_1, b_4]$.
Type 2_M. Type 2_M plays a mixed strategy with support $[b_1, b_3]$ and his payoff is $(\lambda_L + \lambda_M)(v_M + \alpha - b_2)$. If instead he bids $b \in [v_L, b_1)$, then $u_{2M}(b) = \lambda_L\alpha\frac{v_M + \alpha - b}{v_L + \alpha - b}$ [in view of (55)], which is increasing in b since $v_M + \alpha > v_L + \alpha$. Moreover, if 2_M bids $b \in (b_3, b_4]$ then $u_{2M}(b) = (v_H + \alpha - b_4)\frac{v_M + \alpha - b}{v_H + \alpha - b}$ [in view of (58)], which is decreasing in b since $v_M + \alpha < v_H + \alpha$. Therefore type 2_M has no incentive to bid in $[v_L, b_1)$ or in $(b_3, b_4]$.
Type 2_H. Type 2_H plays a mixed strategy with support $[b_3, b_4]$ and his payoff is $v_H + \alpha - b_4$. If instead he bids $b \in [v_L, b_1]$, then $u_{2H}(b) = \lambda_L\alpha\frac{v_H + \alpha - b}{v_L + \alpha - b}$ [in view of (55)], which is increasing in b since $v_H + \alpha > v_L + \alpha$. Moreover, if 2_H bids $b \in (b_1, b_3)$, then $u_{2H}(b) = (\lambda_L + \lambda_M)(v_M + \alpha - b_2)\frac{v_H + \alpha - b}{v_M + \alpha - b}$ [in view of (56) and (57)], which is increasing in b since $v_H + \alpha > v_M + \alpha$. Therefore type 2_H has no incentive to bid in $[v_L, b_3)$.

12.4.2 Step 2: For a small $\alpha > 0$, the inequalities $v_L < b_1 < b_2 < b_3 < b_4$ hold

In the following we use $\Delta \equiv v_M - v_L > 0$ and $t \equiv \frac{1}{\Delta}(v_H - v_M) > 0$. The values of b_1, b_2, b_3, b_4 depend on α, and therefore we write $b_1(\alpha), b_2(\alpha), b_3(\alpha), b_4(\alpha)$. When $\alpha = 0$ we obtain the symmetric setting, with $b_1(0) = v_L$, $b_2(0) = b_3(0) = v_L + \frac{\lambda_M\Delta}{\lambda_L + \lambda_M}$, $b_4(0) = v_L + (\lambda_M + \lambda_H + t\lambda_H)\Delta$. We investigate how b_1, b_2, b_3, b_4 depend on α, for a small $\alpha > 0$, by applying the implicit function theorem to (63)-(65) at $\alpha = 0$, $b_1 = v_L$, $b_2 = b_3 = v_L + \frac{\lambda_M\Delta}{\lambda_L + \lambda_M}$; in this way we obtain $b_1'(0), b_2'(0), b_3'(0), b_4'(0)$. To this purpose we denote the left hand sides of (63),(64),(65) with $f_1(b_1, b_2, b_3, \alpha)$, $f_2(b_1, b_2, b_3, \alpha)$, $f_3(b_1, b_2, b_3, \alpha)$, respectively. Then we obtain

$$
\begin{aligned}
\frac{\partial f_1}{\partial b_1} &= \lambda_M\alpha + (\lambda_L + \lambda_M)(v_M - b_2), \qquad \frac{\partial f_1}{\partial b_2} = (\lambda_L + \lambda_M)(v_L + \alpha - b_1), \qquad \frac{\partial f_1}{\partial b_3} = 0, \\
\frac{\partial f_1}{\partial \alpha} &= -\lambda_M(v_M + 2\alpha - b_1) - (\lambda_L + \lambda_M)(v_L - b_2), \qquad \frac{\partial f_2}{\partial b_1} = 0, \qquad \frac{\partial f_2}{\partial b_2} = -(\lambda_L + \lambda_M)(v_H + \alpha - b_3), \\
\frac{\partial f_2}{\partial b_3} &= (1 - \lambda_H)(v_H + b_2 - 2b_3) + \alpha, \qquad \frac{\partial f_2}{\partial \alpha} = b_3 - \lambda_H(2\alpha + v_M) - (1 - \lambda_H)b_2 \\
\frac{\partial f_3}{\partial b_1} &= -\lambda_L(v_H - b_2), \qquad \frac{\partial f_3}{\partial b_2} = -\lambda_L(v_M - b_1) + (\lambda_L + \lambda_M)(v_H - b_3) \\
\frac{\partial f_3}{\partial b_3} &= (\lambda_L + \lambda_M)(v_M - b_2), \qquad \frac{\partial f_3}{\partial \alpha} = 0
\end{aligned}
$$

We evaluate these derivatives at $\alpha = 0$, $b_1 = v_L$, $b_2 = b_3 = v_L + \frac{\lambda_M \Delta}{\lambda_L+\lambda_M}$ and find

$$\begin{bmatrix} b_1'(0) \\ b_2'(0) \\ b_3'(0) \end{bmatrix} = -\begin{bmatrix} \lambda_L\Delta & 0 & 0 \\ 0 & -(\lambda_L + t\lambda_L + t\lambda_M)\Delta & (\lambda_L + t\lambda_L + t\lambda_M)\Delta \\ -\frac{\lambda_L(\lambda_L+t\lambda_L+t\lambda_M)}{\lambda_L+\lambda_M}\Delta & t(\lambda_L+\lambda_M)\Delta & \lambda_L\Delta \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\frac{\lambda_L\lambda_H}{\lambda_L+\lambda_M}\Delta \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ -\frac{\lambda_L^2\lambda_H}{(\lambda_L+\lambda_M)(\lambda_L+t\lambda_L+t\lambda_M)^2} \\ \frac{\lambda_L\lambda_H t}{(\lambda_L+t\lambda_L+t\lambda_M)^2} \end{bmatrix}$$

Using (62) we see that $b_4'(0) = \frac{(\lambda_L+\lambda_M)\lambda_L\lambda_H t}{(\lambda_L+t\lambda_L+t\lambda_M)^2}$. In next step we use $b_1'(0), b_2'(0), b_3'(0), b_4'(0)$ to derive $\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0}$. However, $b_1'(0)$ does not reveal that $b_1 > v_L$. To this purpose we differentiate (63) twice with respect to α to obtain

$$\lambda_L[2-b_1'(\alpha)]-(\lambda_L+\lambda_M)\{-b_2''(\alpha)[v_L+\alpha-b_1(\alpha)]+2[1-b_2'(\alpha)][1-b_1'(\alpha)]-b_1''(\alpha)[v_M+\alpha-b_2(\alpha)]\} = 0 \quad (66)$$

Evaluating (66) at $\alpha = 0$ yields $(\lambda_L+\lambda_M)[v_M - b_2(0)]b_1''(0) - 2\lambda_M + 2(\lambda_L+\lambda_M)b_2'(0) = 0$, and thus $b_1''(0) = \frac{2}{\lambda_L\Delta}(\lambda_M + \frac{\lambda_L^2\lambda_H}{(\lambda_L+t\lambda_L+t\lambda_M)^2}) > 0$. As a consequence $b_1(\alpha) > v_L$ for a small $\alpha > 0$.

12.4.3 Step 3: Evaluation of $\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0}$

We define $G_i(b)$ as $\lambda_L G_{iL}(b) + \lambda_M G_{iM}(b) + \lambda_H G_{iH}(b)$ for $i = 1, 2$, so that $G(b) = G_1(b)G_2(b)$ is the c.d.f. of the winning bid. In particular, from (51)-(58) we obtain

$$G(b) = \begin{cases} [\lambda_L + \lambda_M G_{1M}(b)]\lambda_L G_{2L}(b) = \lambda_L^2 \frac{\alpha}{v_L+\alpha-b}\frac{v_M-b_1(\alpha)}{v_M-b} & \text{if } b \in [v_L, b_1(\alpha)) \\ [\lambda_L + \lambda_M G_{1M}(b)][\lambda_L + \lambda_M G_{2M}(b)] = (\lambda_L+\lambda_M)\lambda_L \frac{v_M+\alpha-b_2(\alpha)}{v_M+\alpha-b}\frac{v_M-b_1(\alpha)}{v_M-b} & \text{if } b \in [b_1(\alpha), b_2(\alpha)) \\ [\lambda_L + \lambda_M + \lambda_H G_{1H}(b)][\lambda_L + \lambda_M G_{2M}(b)] = (\lambda_L+\lambda_M)\frac{v_M+\alpha-b_2(\alpha)}{v_M+\alpha-b}\frac{v_H-b_4(\alpha)}{v_H-b} & \text{if } b \in [b_2(\alpha), b_3(\alpha)) \\ [\lambda_L + \lambda_M + \lambda_H G_{1H}(b)][\lambda_L + \lambda_M + \lambda_H G_{2H}(b)] = \frac{v_H+\alpha-b_4(\alpha)}{v_H+\alpha-b}\frac{v_H-b_4(\alpha)}{v_H-b} & \text{if } b \in [b_3(\alpha), b_4(\alpha)] \end{cases}$$

Since

$$\begin{aligned} R^F &= v_L G(v_L) + \int_{v_L}^{b_4(\alpha)} b\,dG(b) = b_4(\alpha) - \int_{v_L}^{b_4(\alpha)} G(b)db \\ &= b_4(\alpha) - \int_{v_L}^{b_1(\alpha)} \frac{\lambda_L^2\alpha[v_M - b_1(\alpha)]}{(v_L+\alpha-b)(v_M-b)}db - \int_{b_1(\alpha)}^{b_2(\alpha)} \frac{(\lambda_L+\lambda_M)\lambda_L[v_M+\alpha-b_2(\alpha)][v_M-b_1(\alpha)]}{(v_M+\alpha-b)(v_M-b)}db \\ &\quad - \int_{b_2(\alpha)}^{b_3(\alpha)} \frac{(\lambda_L+\lambda_M)[v_M+\alpha-b_2(\alpha)][v_H-b_4(\alpha)]}{(v_M+\alpha-b)(v_H-b)}db - \int_{b_3(\alpha)}^{b_4(\alpha)} \frac{[v_H+\alpha-b_4(\alpha)][v_H-b_4(\alpha)]}{(v_H+\alpha-b)(v_H-b)}db \end{aligned}$$

we can use this expression to evaluate $\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0}$.

- The derivative of $\int_{v_L}^{b_1(\alpha)} \lambda_L^2 \frac{\alpha[v_M-b_1(\alpha)]}{(v_L+\alpha-b)(v_M-b)}db$ with respect to α is

  $$\lambda_L^2(\frac{\alpha}{v_L+\alpha-b_1(\alpha)}b_1'(\alpha) + \int_{v_L}^{b_1(\alpha)} \frac{[v_M - b_1(\alpha) - \alpha b_1'(\alpha)](v_L+\alpha-b) - \alpha[v_M-b_1(\alpha)]}{(v_M-b)(v_L+\alpha-b)^2}db)$$

  and at $\alpha = 0$ it boils down to 0.

- The derivative of $(\lambda_L+\lambda_M)\lambda_L\int_{b_1(\alpha)}^{b_2(\alpha)}\frac{[v_M+\alpha-b_2(\alpha)][v_M-b_1(\alpha)]}{(v_M+\alpha-b)(v_M-b)}db$ with respect to α is

$$(\lambda_L+\lambda_M)\lambda_L\left(\begin{array}{c}\frac{v_M-b_1(\alpha)}{v_M-b_2(\alpha)}b_2'(\alpha)-\frac{v_M+\alpha-b_2(\alpha)}{v_M+\alpha-b_1(\alpha)}b_1'(\alpha)\\+\int_{b_1(\alpha)}^{b_2(\alpha)}\frac{\{[1-b_2'(\alpha)][v_M-b_1(\alpha)]-b_1'(\alpha)[v_M+\alpha-b_2(\alpha)]\}(v_M+\alpha-b)-[v_M+\alpha-b_2(\alpha)][v_M-b_1(\alpha)]}{(v_M-b)(v_M+\alpha-b)^2}db\end{array}\right)$$

and at $\alpha=0$ it boils down to $\frac{\lambda_M^2(\lambda_L+t\lambda_L+t\lambda_M)^2-2\lambda_L^3\lambda_H}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}$.

- The derivative of $(\lambda_L+\lambda_M)\int_{b_2(\alpha)}^{b_3(\alpha)}\frac{[v_M+\alpha-b_2(\alpha)][v_H-b_4(\alpha)]}{(v_M+\alpha-b)(v_H-b)}db$ with respect to α is

$$(\lambda_L+\lambda_M)\left(\begin{array}{c}\frac{[v_M+\alpha-b_2(\alpha)][v_H-b_4(\alpha)]}{[v_M+\alpha-b_3(\alpha)][v_H-b_3(\alpha)]}b_3'(\alpha)-\frac{v_H-b_4(\alpha)}{v_H-b_2(\alpha)}b_2'(\alpha)\\+\int_{b_2(\alpha)}^{b_3(\alpha)}\frac{1}{v_H-b}\frac{\{[1-b_2'(\alpha)][v_H-b_4(\alpha)]-[v_M+\alpha-b_2(\alpha)]b_4'(\alpha)\}(v_M+\alpha-b)-[v_M+\alpha-b_2(\alpha)][v_H-b_4(\alpha)]}{(v_M+\alpha-b)^2}db\end{array}\right)$$

and at $\alpha=0$ it boils down to $\frac{(\lambda_L+\lambda_M)\lambda_H\lambda_L}{\lambda_L+t\lambda_L+t\lambda_M}$.

- The derivative of $\int_{b_3(\alpha)}^{b_4(\alpha)}\frac{[v_H+\alpha-b_4(\alpha)][v_H-b_4(\alpha)]}{(v_H+\alpha-b)(v_H-b)}db$ is

$$b_4'(\alpha)-\frac{[v_H+\alpha-b_4(\alpha)][v_H-b_4(\alpha)]}{[v_H+\alpha-b_3(\alpha)][v_H-b_3(\alpha)]}b_3'(\alpha)$$
$$+\int_{b_3(\alpha)}^{b_4(\alpha)}\frac{\{[v_H-b_4(\alpha)][1-2b_4'(\alpha)]-\alpha b_4'(\alpha)\}(v_H+\alpha-b)-[v_H+\alpha-b_4(\alpha)][v_H-b_4(\alpha)]}{(v_H-b)(v_H+\alpha-b)^2}db$$

and at $\alpha=0$ it boils down to $\frac{\lambda_H^2\lambda_L(\lambda_L+\lambda_M)t}{(\lambda_L+t\lambda_L+t\lambda_M)^2}+\frac{\lambda_H^2(t\lambda_L+t\lambda_M-\lambda_L)^2}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}$, which is equal to $\lambda_H^2\frac{(\lambda_L+\lambda_M)^2t^2+\lambda_L^2}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}$.

Therefore

$$\begin{aligned}\left.\frac{dR^F}{d\alpha}\right|_{\alpha=0} &= \frac{(\lambda_L+\lambda_M)\lambda_H\lambda_L t}{(\lambda_L+t\lambda_L+t\lambda_M)^2}-\frac{\lambda_M^2(\lambda_L+t\lambda_L+t\lambda_M)^2-2\lambda_L^3\lambda_H}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}-\frac{(\lambda_L+\lambda_M)\lambda_H\lambda_L}{\lambda_L+t\lambda_L+t\lambda_M}\\ &\quad -\lambda_H^2\frac{(\lambda_L+\lambda_M)^2t^2+\lambda_L^2}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}\\ &= \frac{-(\lambda_H^2+\lambda_M^2)(\lambda_L+\lambda_M)^2t^2+2\lambda_L(\lambda_L+\lambda_M)(\lambda_H^2-\lambda_M^2)t-\lambda_L^2(1-\lambda_L)^2}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}\end{aligned}$$

12.4.4 Step 4: $\left.\frac{d(R^S-R^F)}{d\alpha}\right|_{\alpha=0}>0$

It is straightforward to see that

$$\begin{aligned}R^S &= \lambda_L v_L+\lambda_M\lambda_L(v_L+\alpha)+\lambda_M(\lambda_M+\lambda_H)v_M+\lambda_H\lambda_L(v_L+\alpha)+\lambda_H\lambda_M(v_M+\alpha)+\lambda_H^2v_H\\ &= v_L+((\lambda_M+\lambda_H)^2+\lambda_H^2t)\Delta+(\lambda_H\lambda_L+\lambda_L\lambda_M+\lambda_H\lambda_M)\alpha\end{aligned}$$

and thus $\frac{dR^S}{d\alpha}=\lambda_H\lambda_L+\lambda_L\lambda_M+\lambda_H\lambda_M>0$. The inequality $\left.\frac{d(R^S-R^F)}{d\alpha}\right|_{\alpha=0}>0$ is equivalent to

$$\lambda_H\lambda_L+\lambda_L\lambda_M+\lambda_H\lambda_M+\frac{(\lambda_H^2+\lambda_M^2)(1-\lambda_H)^2t^2-2\lambda_L(1-\lambda_H)(\lambda_H^2-\lambda_M^2)t+\lambda_L^2(1-\lambda_L)^2}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}>0$$

After suitable manipulations, the left hand side of this inequality is written as $\frac{(1-\lambda_L^2)(\lambda_L+\lambda_M)^2}{2(\lambda_L+t\lambda_L+t\lambda_M)^2}[t-\frac{\lambda_L(2\lambda_H^2+\lambda_L^2-1)}{(1-\lambda_L^2)(\lambda_L+\lambda_M)}]^2+\frac{\lambda_H\lambda_L^2[2\lambda_H(1-\lambda_L^2-\lambda_H^2)+\lambda_M(1-\lambda_L^2)]}{(\lambda_L+t\lambda_L+t\lambda_M)^2(1-\lambda_L^2)}$, which is positive.

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