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Incomplete altruistic preferences

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Abstract

This note is concerned with qualitative restrictions on individual binary preferences of decision makers over profiles of individual actions to account for intersubjective sensitivity to welfare, or preferences, of others, when numerical representations by utility functions fail, due to incompleteness of the binary relations. The restrictions are to accommodate notions of altruism, egoism, paternalism, and others, as externalities of private actions of others through others' welfare, or preferences, as in the literature, e.g. [11].

A conclusion is suggested to the exent that a notion of utility which is independent of numerical representations of preferences may be necessary to deal with attitudes of concern towards the preferences of others.

Keywords: incomplete preferences, utility, altruism, egoism, paternalism, externalities. JEL class. codes.: D01, D62, D64.

1 Introduction

Assume a standard set-up of intersubjective decision theory where the objects of individual preferences consists solely of profiles of individual actions. Let $\mathcal{I}$ be a finite set of decision makers, $i \in \mathcal{I}$, each of whom decides over a finite[1] set of exhaustive alternative private actions X_i, $x_i \in X_i$. Let $X = \times_i X_i$, the Cartesian set of states of the economy, which are the objects of preference for each individual, $x \in X$. It will be convenient to denote the state x in the usual way, (x_i, x_{-i}), where $x_{-i} \in X_{-i} = \times_{k \neq i} X_k$.

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[1] We wish to concentrate over incompleteness of the preference relation, and finiteness casts away topological reasons for the lack of numerical representability.

Assume that the individual own components of the states, and objects of decision, are own acts of consumption. Then, two different states with the same profile of acts of consumption of others may be valued differently by a subject only because of different own acts of private consumption; and two different states with the same own act of private consumption may be valued differently only because of externalities due to different profile of consumption of the others. The structure is capable to accommodate, with no restrictions, all different cases of private consumption and of externalities over the consumption of others. Consequently, any restrictions on individual preferences to accommodate for "intersubjective sensitivity" to the welfare, or preferences, of others would consist in constraints on preferences over private consumption, or, more likely, admissible externalities.

A convenient way to implement such restrictions is formally similar to the theory of imperfect information in Bayesian decision theory, and it consists in making different states, of profiles of actions, "value indiscernible". To this end, one may stipulate that the subject's evaluation of states depends on own private consumption, and on others' *evaluations* of states, rather than others' profiles of actions. Thus, two different states may be characterized by the same own consumption, and the same profile of evaluations of others, so that the subject's evaluation may therefore be assumed to be invariant with respect to the others' profile of acts of consumption.

If the subject's preference are represented by an utility function u_i, as we will see in more detail, and before further generalization, the restriction consists in the postulation of the existence of a real function φ such that $u_i(x) = \varphi(x_i, u_{-i}(x))$, with usual notation, and which may separate the evaluation of states with respect to own consumption, and with respect to others' evaluations of the entire profile of acts of consumption.

Now, the subject's preferences, given a constant own act of consumption, are no longer fully free to reflect all externalities on others' acts of consumption, but only so long as these are mediated by others' evaluations, and these perform formally like a reliable "signal of welfare": the subject's preferences are sensible only to others' evaluations, and only indirectly to others' acts of consumption.

One may interpret this formal structure, and consider what consequences it may have on the nature of the states of the economy. In fact, the structure is compatible with alternative interpretations of the ontology of the states.

One may consider the individual evaluations not to be real parts of the states, but consist in external binary comparisons of them. Under this view, intersubjective sensitivity of individual preferences still concern externalities regarding others' acts of consumptions (and do not concern others' "welfare", which is not part of the states). Such externalities, on the other hand, are "blurred" by the signals of the others' comparisons: in the case of indifference, typically, the signal makes the states be "value indiscernible".

Alternatively, on may view the subject's preference relation as a comparative measure of own welfare at the states, and this to be parts of the states. These, therefore, may be viewed as profiles of acts of consumptions (which are, possibly, strictly private, and with no externalities), and related "acts of welfare", in a way which may resemble the specification of an input-output profile in the identification of a technology. Thus, restrictions on individual preferences may remain independent on others' acts of consumption, and depend instead on the welfare which others derive from them. In cases of indifference, the states are then really "welfare equivalent".

In sum, the intersubjective sensitivity we are considering is independent of the ontology of the states, and, therefore, does not distinguish consumption externalities from concern over others' welfare.

We intend to generalize this formal structure to the case of weakenings of the axiomatization of individual preferences which do no longer allow for numerical representations. This generalization may be justified by the plausibility of examples of preference relations which cannot be represented numerically, due to incompleteness, and gives an opportunity for a more articulated discussion of intersubjective relations between preferential attitudes, when these include strict preference, indifference, and also incomparability.

2 Preliminary material

Let $P_i \subset X^2$ be the binary "strict" preference relation of subject i, which is assumed to be asymmetric (hence irriflessive) and transitive, and to be interpreted in the usual way: $(y, x) \in P_i$, when i prefers y to x.

> It is possibly more common to axiomatize a weak preference relation R_i of "not worse than". The two relations may naturally be made to coincide by rules of one-to-one correspondence which restrict the axiomatizations equivalently: for example, $(y, x) \in R_i$ if and only if $(y, x) \notin P_i$ (see, e.g., [8] pp.21–24). But the strict relation is useful in the study of incomplete

preferences because attitudes of indifference and incomparability are more easily separated form strict preferences, as in what follows.

Let $P_i^{-1} = \{(x,y) \mid (y,x) \in P_i\}$ be the inverse of P_i. Let $G_i = P_i \cup P_i^{-1}$, be the binary relation of judgement: $(y,x) \in G_i$ when i is in capable of expressing a judgement of "strict" preference between the states. If $x \in X$, define $P_i(x) = \{y \mid (y,x) \in P_i\}$ to be the set of states which are better than x according to P_i, and $P_i^{-1}(x) = \left\{y \mid (y,x) \in P_i^{-1}\right\}$ the set of states worse than x. Define also $G_i(x) = \{y \mid (y,x) \in G_i\}$ to be the set of states which can be judged against x under strict preference.

Now, let $N_i = X^2/G_i$ be the "residual" relation with respect to judgement; clearly, $(y,x) \in N_i$, when i does not judge the states under strict preference. N_i is reflexive and symmetric, but not necessarily transitive. If N_i happens to be transitive, then it is an equivalence relation which partitions X into so called equivalence classes; these can be put into the linear order defined by the P_i between representative elements of the classes: the preference relation P_i is then said to be complete[2]. Given that X is here assumed to be finite, completeness of P_i is a necessary and sufficient condition for regularity, i.e., for the existence of a numerical representation by a real function $u_i : X \to \mathbb{R}$, which takes the traditional name of utility function.

Via numerical representation, if $(y,x) \in N_i$, then $u_i(y) = u_i(x)$, and the equality is interpreted as an "equal welfare evaluation" of the states on behalf of subject i, who thus values them as *indifferent*.

Objections against the axiom of transitivity of P_i are set aside here. These are often based on issues concerning the cumulative results of differences on the ontology of states along a chain of pairs of strict preferences which may pose a problem on the judgement, weather on matters of fact, or of their perception. (See, e.g., [1].)

Similarly, we will disregard objections against transitivity as a condition for indifference (see, e.g., [5]). These often rest on assumptions concerning "perception thresholds", whose cumulation may snap the chain of evaluations. In fact, we will retain transitivity as a necessary condition for indifference in order to view it as an evaluation of "equal welfare". An alternative theoretical route of indifference as an evaluation of "similar welfare" might be

[2] Mathematical completeness of an order relation usually refers to existence of suprema; but here we are following microeconomic terminology.

grounded on some notion of "distance" between states: there, transitivity may be guaranteed only in some neighborhood of the states.

On the other hand, if P_i is not complete, a restriction of N_i may always be defined to guarantee transitivity, and obtain a relation of equivalence which may be interpreted as indifference. Weather or not a numerical representation is possible, it will be reasonable to assume that the subject may be capable, at least in some cases, to draw an evaluation of indifference between pairs of states, and that the evaluation respects transitivity on the ground of the philosophy of "equal welfare".

For example, it may be reasonable to assume that, for any x, x is indifferent to itself, so that identity implies indifference. This suffices to guarantee that the relation of indifference to be reconstructed is never empty.

2.1 Equivalence relations and partitions on X

Equivalence relations $Q_i \in \mathcal{Q}_i$ on X are one-to-one with partitions into classes of equivalence (or "blocks") thereby induced on X. Let us denote in bold $\mathbf{Q}_i$ the partition induced by the equivalence relation Q_i. Thus, $\mathbf{Q}_i^{\min}$ (the same for all i), for example, will denote the "fine" partition: $\{\{x\} \mid x \in X\}$, and $Q_i^{\min}$ the corresponding smallest equivalent relation: $\{(x, x) \mid x \in X\}$. If $x \in q \in \mathbf{Q}_i$, $q_i(x)$ denotes the block x belongs to, according to $\mathbf{Q}_i$. Hence, if $q_i(x) \in \mathbf{Q}_i^{\min}$, $q_i(x) = \{x\}$.

Partitions on X are in a partial order of refinement and coarsening. If $\mathbf{Q}_i$ is a refinement of $\mathbf{Q}'_i$ (hence, the latter is a coarsening of the former), then any $q' \in \mathbf{Q}'_i$ is partitioned by elements of $\mathbf{Q}_i$. The order has a minimum $\mathbf{Q}_i^{\min}$ and a maximum $\mathbf{Q}_i^{\max} = \{X\}$[3]. The set of partitions is a complete lattice [2]. On the other hand, if $Q_i^{\min}$ is an acceptable transitive restriction of N_i, $Q_i^{\max}$ may not be so, and in fact there may not be an acceptable maximum transitive restriction.

The partitions which may be considered consistently with P_i, are indeed required to satisfy the constraint that if $y \in q_i(x)$, then $(y, x) \in N_i$. Hence, it may not be that, if $y \in q(x)$ then $P_i(x) \cap P_i^{-1}(y) \neq \varnothing$; for, otherwise, if $z \in P_i(x) \cap P_i^{-1}(y)$, then absurdly $y \in P_i(x)$, by transitivity of the P_i. This amounts to a strong constraint on acceptable partitions, which may

[3]This is the other way round with respect to standard pratice: we prefer to consider the fine partition as the minimum because we take it to be the guaranteed bottom line of any relation of indifference.

be called non-separation.[4]

On the other hand, the converse does not hold: in fact, the transitive restriction of N_i we are after, may allow that $y \notin q_i(x)$ and $(y,x) \in N_i$.

A common criterion for a transitive restriction of the residual relation is the principle of *equi-disposition*. Two states x and y are equi-disposed when $P_i(x) = P_i(y)$ and $P_i^{-1}(x) = P_i^{-1}(y)$. Then, necessarily, $(y,x) \in N_i$, for otherwise, say, $(y,x) \in P_i$, hence $y \in P_i(x)$, when in fact $y \notin P_i(y)$.

If x and y are equi-disposed, and so are y and z, then x and z are also equi-disposed. Hence $(x,y) \in N_i$ and $(y,z) \in N_i$ imply $(x,z) \in N_i$: that is, equi-disposition implies transitivity. Define $E_i = \{(y,x) \in N_i \mid P_i(x) = P_i(y) \text{ and } P_i^{-1}(x) = P_i^{-1}(y)\}$ to be the relation of equi-disposition. Clearly, $E_i \subseteq N_i$.

The relation (and induced partition) of equi-disposition is not the only equivalence relation which is equi-disposed, i.e., where any two elements of the same block are equi-disposed; for example, $\mathbf{Q}_i^{\min}$ is equi-disposed also. $\mathbf{E}_i$ is the maximum equi-disposed partition (it exists, and it was defined): here, any two elements belonging to different blocks are never equi-disposed. Any other equi-disposed partition will be a refinement of $\mathbf{E}_i$.

If, for any $(y,x) \in N_i$ states x and y are equi-disposed, then the residual N_i is transitive. The converse obviously also holds, as is immediate from the possible numerical representation: if N_i is transitive, then $u_i(x) = u_i(y)$, and any state preferred to x is also preferred to y, any state worse than x is also worse than y. In other words, in the case of regular preferences equi-disposition is identifiable with a transitive N_i, i.e., in the interpretation, with indifference.

On the other hand, if preferences are incomplete, then equi-disposition is only a sufficient criterion for a transitive restriction of the residual relation. Other similarly sufficient criteria for transitive restrictions can be considered. For example, both upper equi-disposition, i.e., sameness of better states, and lower equi-disposition, i.e., sameness of worse states, guarantee transitivity. Hence, one may define, respectively, $E_i^U = \{(y,x) \in N_i \mid P_i(x) = P_i(y)\}$, and $E_i^L = \{(y,x) \in N_i \mid P_i^{-1}(x) = P_i^{-1}(y)\}$. Obviously, $E_i \subseteq E_i^U$, $E_i \subseteq E_i^L$ and $E_i^U \cap E_i^L = E_i$.

[4] A partition $\mathbf{Q}_i$ of X is non-separated when, for any two blocks q and q', any state z of q' is never "between" states x and y of q, in the obvious sense of the order P_i.

Non-separated partitions may be compared to non-crossing non-nesting partitions, with respect to the non-linear order P_i, and disregarding relations between elements in the same block. With these caveats, the two coincide. See for definition, e.g., [4].

Similarly to $\mathbf{E}_i$, both $\mathbf{E}_i^U$, $\mathbf{E}_i^L$ are maxima in their respective criteria: any respective refinement enjoys the property of upper, respectively, lower equi-disposition.

Other criteria can be conceived. If respect of transitivity characterizes a set of pairs in the residual relation, and it is, for this reason, to be considered as an indifference relation, then the relation of indifference is underdetermined.

Example 1 *Let* $X = \{x, y, z\}$*, and let* $P_i = \{(y, x)\}$*. Then,* $G_i = \{(y, x), (x, y)\}$*, e* $N_i = \{(x, x), (y, y), (z, z), (x, z)(z, x), (y, z), (z, y)\}$*.*

There follows: $E_i = \{(x, x), (y, y), (z, z)\}$*;* $E_i^U = \{(x, x), (y, y), (z, z), (y, z), (z, y)\}$*;* $E_i^L = \{(x, x), (y, y), (z, z), (x, z), (z, x)\}$*.*

> The example can be interpreted to show the plausibility of incomplete preference relations. Let x be the state: "I enjoy a red Ferrari" (other things being equal); let y be the state: "I enjoy a yellow Ferrari"; let z be the state: "I invest the money in a mansion". Then it is plausible that "I" have a definite taste for the color of my prospective sports car, and be unconvinced weather to spend my money on that, or on a house.

Example 1 shows that the relation of equi-disposition is never empty, and contains at least (the partition of equi-disposition is refined by) the minimum equivalence relation (the fine partition), as above, whereby each state is in a relation of indifference with, and only with, itself.

The example also shows that equivalence relations which are consistent with P_i in general do not have a maximum. In the example, both E_i^U and E_i^L are maximal elements in the family.

Partition $\mathbf{E}_i$ is notable because its classes can be put into the same partial order as there exists between representative elements of the classes according to P_i, and it is the maximum partition with this property: any refinement of $\mathbf{E}_i$ (any other equi-disposed partition) enjoys the property; any coarsening of it does not.

2.2 Maximal states and optimization

If $Y_i \subseteq X$, and $P_i(x) \cap Y_i = \varnothing$, then x is called a maximal state (according to P_i) in Y_i, or a maximal state *tout court* when $Y_i = X$. Existence of maximal elements is guaranteed here by the finiteness of X. Let $M_i^{Y_i}$ the set of maximal elements for i in Y_i (M_i, *tout court*). Qualify $\mathbf{Q}_i$ to be equi-maximal if, for any, $x \in M_i$, $q_i(x) \subseteq M_i$, and call $q_i(x)$ a maximal block. If

$\mathbf{Q}_i$ is equi-maximal, then a subset of its blocks partitions M_i: no block of partition $\mathbf{Q}_i$ may contain both maximal and non-maximal elements. The maximal block $q_i(x)$ will be unique when $q_i(x) \supseteq M_i$.

If preferences are complete, the maximal block will be unique *a fortiori*, and if two states x and y are not in the same block then they are ordered by strict preference, and at least one is not maximal. But if preferences are not complete, x and y may be not ordered, and be both maximal states.

$\mathbf{E}_i^L$ is not in general equi-maximal, as in the case of example 1, but $\mathbf{E}_i^U$ is so: if $P_i(x) = \varnothing$ and $y \in q_i(x)$, then $P_i(y) = \varnothing$, because of upper equi-disposition. In $\mathbf{E}_i^U$, also, the maximal block is unique.

Define *optimization* to be the relation between a structure of preference (P_i, Q_i) on X and the implementation of a state in $Y_i \subseteq X$, on behalf of i.

Set Y_i is to formalize "objective" constraints to the optimization: these may be "external", like a consumer's budget, or "internal", like the subject's beliefs over others' choices in a strategic setting. Standardly, the subjective structure of preference is maintained to be independent of, i.e., invariant with, the constraint.

At least two different kinds of optimization procedure can be conceived of: a "static", or "external", *choice* procedure, and a "dynamic", or "internal", *substitution* procedure. A notion of "rationality" may be employed to qualify the relation between decisions and preferences, both in the case of choice decision and of substitution decision.

A choice will be *rational* when the state selected is maximal; otherwise, it will be *irrational*.

On the other hand, a substitution will be rational when state x is replaced by a better state, or by an indifferent one.

Instead, one may qualify a substitution to be rational only when x is replaced by a better state, and, if x is maximal, with "nothing else"; then, the "non-substitution" of maximal x is to be considered as not a substitution. In fact, the non-substitution of x is here the (rational) substitution of x with an indifferent state, i.e., x itself. Consequently, in our setup, any state may always be substituted, and the process of dynamic optimization may not have a temporal end, but only reach a steady state.

> The present postulation is indeed in line with the tradition of strong Pareto improvements, where all substitutions, including indifference states, are considered to be individually rational.

Unlike choice, substitution is then qualifiable in more than two ways: non-rationality is spelled out into irrationally (the substitution with a worse state), and a-rationality (substitution of a state by an incomparable state).

One may define x's "welfare" by its block $q_i(x)$. Then, the two optimization procedures will be welfare equivalent if and only if the rationally chosen maximal state's welfare is the same as the welfare "eventually" reached through rational substitution. In order to guarantee this correspondence, $\mathbf{Q}_i$ must be equi-maximal. Otherwise, if, for example, $\mathbf{Q}_i$ is not upper equi-disposed, and therefore possibly is not equi-maximal, a substitution process reaching the chosen maximal state, or other equivalent state, may further proceed to higher welfare level, as is the case in example 1, where welfare $q_i(y)$, higher than welfare $q_i(x)$ of maximal x, may be reached rationally when the indifference partition is $\mathbf{E}_i^L$.[5]

3 Intersubjective preferences

Scapparone [11] defines a subject i's individual preferences, which are represented numerically by an utility function u_i, to be, in our terminology, intersubjective when there exists a function φ_i such that

$$u_i(x) = \varphi_i[x_i, u_{-i}(x)] \tag{1}$$

in a standard notation, and i's utility over states is separable into an argument of "private consumption", and arguments of the utilities that others "derive" from the states.

By this formulation, subject i may be qualified as *benevolently*, respectively, *malevolently altruistic*, when φ_i increases, respectively, decreases in the latter arguments; subject i is qualified to be *egoistic* in the case of "neutrality", i.e., of a constant φ_i in the same arguments.

One can show ([11], p.12) that the existence of φ_i in (1) is equivalent to the condition

$$\Delta_i \cap (\cap_{k \neq i} E_k) \subseteq E_i \tag{2}$$

where $\Delta_i = \{(y,x) \mid y_i = x_i\}$ is the i-diagonal of X. Clearly, here $E_i = N_i$.

[5] On similar issue, e.g., [9]

It is indeed plain that if, in (2), given the same own private consumption of i, x_i, all other subjects are indifferent between two different states x and x', then all independent variables in (1) are constant, and i's utility must be the same for any existing φ_i, i.e., i is indifferent in (2): hence (1) implies (2). For the converse, assume that, under the same assumption, i.e., $u_{-i}(x) = u_{-i}(x')$ and constant own private consumption x_i, yet $u_i(x) \neq u_i(x')$; then 2 fails also.

Given the Cartesian structure of X, Δ_i is the set of all pairs whose elements lie on the same "row" of subject i. As a shorthand notation, we may write $(y,x)^{\Delta_i}$ to mean $(y,x) \in \Delta_i$.

In the interpretation, condition (2) requires that, given a constant own private consumption, two different states of the economy must be indifferent to subject i if they are indifferent to all other subjects.

Condition (2) is satisfied trivially when $\Delta_i \cap (\cap_{k \neq i} E_k) = \varnothing$, i.e., when pairwise comparisons between states with the same private consumption of i do not exhibit indifference for some other subject. This is the possibly counterintuitive case of intersubjective preferences of subject i, independently of what preferences subject i has.

$(\cap_{k \neq i} E_k) \subseteq E_i$ is equivalent to $E_i^C \subseteq (\cap_{k \neq i} E_k)^C = \cup_{k \neq i} E_k^C$; but if preferences are complete, E_i and G_i are complements in X^2. Hence, (2) turns out to be equivalent to $G_i \subseteq \Delta_i^C \cup (\cup_{k \neq i} G_k)$, i.e., to

$$\Delta_i \cap G_i \subseteq \cup_{k \neq i} G_k \tag{3}$$

Therefore, subject i's preferences are intersubjective if and only if, given some states x and x' with the same own private consumption, if i holds a strict judgement between the states, then someone else also holds a strict judgement between the two.

Again, (3) is trivially satisfied when $\Delta_i \cap G_i = \varnothing$. But now the trivial satisfaction of the condition has an intuitive interpretation: this is the case of a subject whose egoistic preferences do not depend on others', but only on own private consumption.

As it is suggested in [11], p.13, (2) may be used as a qualitative definition of intersubjective preferences (that is, in cases of no numerical representation), and so is condition (3); but the two are equivalent only if preferences are complete. When they are not, therefore, the conditions provide different definitions of intersubjective preferences, and one may examine such differences.

On our way now to considering incomplete preferences, it is first of all necessary to reformulate condition (2), and dispose of the implicit assumption that the relations of indifference be the relations of equi-disposition E_i, E_k. Hence, assume Q_i to be an unspecified equivalence relation on X consistent with P_i , and consider the analogue condition:

$$\Delta_i \cap (\cap_{k \neq i} Q_k) \subseteq Q_i \tag{4}$$

If condition (3) is now deployed as a definition of intersubjectivity in the case of incompleteness, one may distinguish within a variety of different intersubjective attitudes.

First, preferences will be *strictly intersubjective* (the analogue of benevolence/malevolence in the case of regularity) when $(y,x)^{\Delta_i} \in G_i$ *sub* (3). Second, preferences will be *weakly intersubjective* in the trivial case of $\Delta_i \cap G_i = \varnothing$; then, $(y,x)^{\Delta_i} \in N_i$. Weak intersubjectivity, then, can be distinguished according to weather $(y,x)^{\Delta_i} \in Q_i$, and the pair is indifferent, or $(y,x)^{\Delta_i} \in U_i = N_i/Q_i$, i.e., the case of incomparability, or "lack of judgment" between pairs in Δ_i. The distinction allows for the definition of two possibly polar different attitudes towards others: a "sturdy" egoistic attitude, in line with the case of regular preferences; and an opposite *confused* attitude, when the subject is deranged by others towards a failure of judgement, in spite of not there being any issue concerning his own private consumption.

Let us illustrate the work done by conditions (3) and (4) in the case of only two subjects, i and j, to highlight points of consonance and discrepancy. Let $(y,x)^{\Delta_i}$ be a pair of states with constant private consumption of subject i. The subject's preferences will, and will not, be intersubjective, according to the two definitions, when the pair belongs to binary relations of i and j, as specified in the following table, with obvious notation:

	G_j	Q_j	U_j
G_i	3, 4	¬3, ¬4	¬3, 4
Q_i	3, 4	3, 4	3, 4
U_i	3, 4	3, ¬4	3, 4

(5)

It appears that the conditions agree upon the definition of egoistic i, and on (strict and weak) i's intersubjectivity by default, i.e., whenever $\acute{j}$'s evaluation of the pair is a strict preference. They also agree on accepting intersubjectivity when both i and j are confused, and agree on rejecting intersubjectivity when i has a strict preference and j is indifferent.

Finally, the two cases of interest, where the criteria disagree, concern, not unexpectedly, incomparability. If i is confused when j is indifferent, (3) stipulates intersubjectivity and (4) denies it; vice-versa, whenever i holds a strict preference when j is confused.

Axiomatically, the table shows how the criteria are neither stronger nor weaker one of another; this can be seen easily also by considering that (4) is equivalent to

$$\Delta_i \cap (G_i \cup U_i) \subseteq \cup_{k \neq i} (G_k \cup U_k) \tag{6}$$

and by comparing the requirement thus reformulated with (3).

The taxonomy in table 5 spells out differently in relation with the selected relation of indifference. For example, if $Q_j = Q_j^{\min}$, then the case $\left(U_i, Q_j^{\min}\right)$ is empty, for the only pairs $(y,x)^{\Delta_i} \in Q_j^{\min}$ require that $x = y$, hence it also holds that $(y,x)^{\Delta_i} \in Q_i$.

3.1 Functional separability, "force" of evaluation, and paternalism

In the case of regular preferences, standard terminology views the relation of indifference as the transitive relation which is "residual" with respect to the relation of strict judgement, but the two are in fact symmetrically only complements to each other. Indeed, the formal structure does not justify that view more that the converse one, i.e., that G_i is the residual relation with respect to E_i. No elements of the internal structure of judgement are given to provide for any formal backing to paradigmatic statements of the kind of: «I have no reason to prefer this to that, *hence* I am indifferent».

On the other hand, one might introduce explicitly some sort of "Keynesian" notion of a "force of the evaluation", in order to represent, in a reduced form, the order of residuality which we have assumed there to be: indifference is residual because it is intrinsically "weaker" that strict judgement.

Arguments in favour of this thesis, as well as of the opposite one, i.e., that an evaluation of indifference is "more forceful" than a strict judgement over the same states, can be substained, but they are not directly at issue here. Our justification for considering strict judgement as "more forceful" lies only in the assumption of functional separability of utility; the postulation suggested here simply allows for a more general treatment of what may be equivalent to functional separability under non regular preferences, that is, as the requirement of "no paternalism" embedded in the paradigmatic statement: «I have no motives, other than my own private consumption, than consideration of your motives». Therefore, we take the judgement of strict preference, which is not allowed, by functional separability, to be a "reply" to indifference, to be "stronger" than indifference, in a different sense than that of its inner bigger "force" with respect to indifference on behalf of the same subject: to repeat, the interpretation is that one's own strict preference would be based on reasons concerning others' consumption that go beyond the reasons of the others; in a word, paternalistic reasons. Hence, lack of paternalism is in the end formally expressed in terms of strength: «With a given constant own private consumption, my evaluation cannot be stronger than yours.»

This relative "strength" in the structure of judgements, or lack thereof, is also necessary to separate cases within the role played by the "equality of utility", when this is no longer the residual of G_i, and the residual is articulated into the separate relations Q_i and U_i.

To this purpose, the "force" of a lack of evaluation, i.e., of the incomparability relation U_i, must be specified: but this can only be less that the force of the other two, under common sense. Now, the requirement of no paternalism to constrain intersubjectivity will require that subject i's evaluation cannot be stronger than that of the other subjects in the case of incomparability as well, as illustrated by the paradigm: «Given an equal own consumption of mine, if you are not concerned (you make no comparison), why should I?»

> Attention should be paid to the different criteria which are used to justify the relative strength of strict judgement vs. indifference, and that of incomparability vs. the other two. In the latter case, it is easy to make assumptions on the inner structure of the judgement; in the former, the case was tackled by interpreting the assumption of functional separability directly as a requirement of no paternalism[6]. One could object that this requirement of no paternalism, therefore, is carried over from the first to the second case on a different ground, and illegitimately; moreover, in the first case, considerations are made over the comparison between *own* strict judgement and *others'* indifference, whereas in the second case relative strength is considered in the comparison between one's *own* judgement and one's *own* lack thereof. But we do not claim that the issue of paternalism is treated in the two cases by the same criteria. In fact, we do have one *a priori* criterion for the second case (the "force" of the judgement vs. the "weakness" of the lack of it), and a "Bayesian"-like reinterpretation (cfr., supra, Introduction) of the constraint implied by functional separability.

In table 7 (as indeed in table 5) relations are listed in decreasing order of strength by

[6] Our view of paternalism is standard in decision theory: see, e.g., [7]. Philosophical literature on paternalism is not always crisp in the distinction between concern towards others' "objective" welfare, and concern over their "interests and values", as in [3].

row, from top to bottom, and by column, from left to right, and cases of paternalism are strictly above the main diagonal; in the other cases intersubjectivity is guaranteed with no paternalism. In a Boolean spirit, one may derive an algebra of notions from the set theoretical axiomatization; then, the complement of paternalism may be named “maternalism”, in order to denote a sensitivity to others’ welfare which does not surmount “consumer’s sovereignty” to make some criterion of “objective” welfare prevail.[7]

	G_j	Q_j	U_j
G_i	yes	no	no
Q_i	yes	yes	no
U_i	yes	yes	yes

(7)

Maternalism obviously agrees with (3) and (4) in the cases where neither one’s own, nor others’ incomparability is involved, and, similarly, in the cases where one’s own incomparability does nor reply to others’ indifference. It is interesting to observe that it agrees with criterion (3) when this disagrees with criterion (4).

Finally, maternalism disagrees with both other criteria on egoism as intersubjective by default; in fact, (Q_i, U_j) is a notable case of paternalistic egoism: «Given my constant own private consumption, I introduce a judgement of indifference between states which cannot be justified by your lack of judgement». Paternalistic egoism turns out not to be “qualitatively separable,” unlike flat egoism with regular preferences.

Comparison of the two tables shows that maternalism is axiomatically stricter than (3): the cases admitted by the former make a subset of those allowed for by the latter. On the other hand, (3) is clearly a partial requirement of no paternalism; in order for the two requirements to be made equivalent, it is necessary and sufficient to conjoin the following additional requirement to (3):

$$\Delta_i \cap Q_i \subseteq \cup_{k \neq i} (Q_k \cup G_k) \tag{8}$$

as it is easy to show.

On the other hand, maternalism is neither weaker, nor stronger than (4).

[7] Obviously, our use of the notion of “maternalism” does not coincide with common use in other areas, where, roughly, it is juxtaposed to authoritarianism.

4 Diagrams

The criteria considered so far can be applied to generate diagrams in the Cartesian structure of the states.

In figure 1, there are only subjects i and j, and the diagram is generated by the postulation of (4).

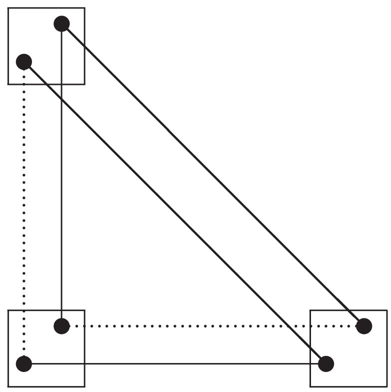

Figure 1: Q_i and Q_j under $\Delta_i \cap (\cap_{k \neq i} Q_k) \subseteq Q_i$

In the figure, the squares are three states (not necessarily all distinct) in a Cartesian setting, i.e., with a constant private consumption of i along a row, and of j down a column. Following "game-theorethic" graphic conventions, pairs of (qualitative) "payoffs" for i and j are inside each square, respectively down left, and up right. Segments join states which are indifferent to the subjects. Dotted segments are specific sufficient postulates, and solid segments parallel to these are respective immediate consequences by axiom (4). Diagonal segments follow from transitivity of the equivalence relation.

On the other hand, condition (3) does not generate a diagram for any relation. Moreover, (3) is equivalent to

$$\Delta_i \cap (\cap_{k \neq i} N_k) \subseteq N_i \tag{9}$$

and indeed relation $N_i = Q_i \cup U_i$ does not enjoy transitivity.

Maternalism does not generate a triangle either, because again transitivity is not guaranteed. Unlike the previous case, however, the reply to incomparability is strictly constrained by the requirement of monotonicity, as if in a fixed point. This allows for the construction of the rectangular diagram in figure 2, with the conventions of figure 1, except that segments now pair incomparable states.

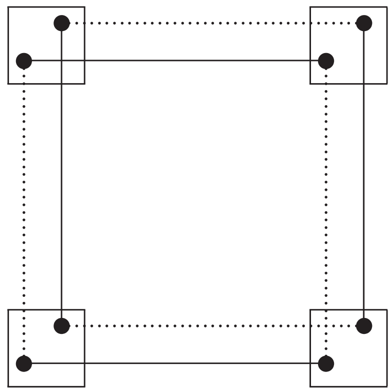

Figure 2: Maternalism on U_i and U_j

4.1 Nash equilibria

Considering its Cartesian structure, the set X of states of the economy may be regarded as the set of ("pure") strategic profiles in a normal form game; this game may have "qualitative" Nash equilibria, i.e., with no recourse to numerical payoffs. We shall proceed mainly by way of examples.

Let us remain within the simplifying assumption of a two player game. Let $B_i : X_j \longrightarrow X_i$ be i's correspondance of best reply to j, so that $x_i \in B_i(x_j)$ when (x_i, x_j) is maximal in the set $\{(x'_i, x_j) \mid x'_i \in X_i\}$. Similarly, $B_j : X_i \longrightarrow X_j$. As usual, then, $\left(x_i^*, x_j^*\right)$ is a Nash equilibrium if, simultaneously, $x_i^* \in B_i(x_j^*)$ and $x_j^* \in B_i(x_i^*)$. Existence of Nash equilibria is obviously not guaranteed.

The various postulations concerning intersubjective preferences define restrictions on the qualitative "payments" in the game, hence on the best reply correspondences, and on possible equilibria.

If a Nash equilibrium is not strict on i's side, say, then $B_i(x_j)$ is not a singleton ([10], p.50). Standardly, then, the elements in $B_i(x_j)$ are indifferent to each other; but in our setup, the maximal states of the set $\{(x'_i, x_j) \mid x'_i \in X_i\}$ may also be incomparable. Either case plays the role of specific postulates in figures 1 and 2.

Example 2 to follow is a case of a type which may be made more general, and concern the case of numerical representations.

Example 2

	j_1	j_2
i_1	*1,2*	a,b
i_2	*1,2*	*1,2*

(10)

In the game of table 10, (i_2, j_1) *is a Nash equilibrium which is weak "twice" because neithey best reply correspondance is a singleton in the equilibrium point. Because of numerical representations in payoffs, any player's best replies are indifferent to each other, and this constrains the other player's payoffs from states with a constant private consumption of the other, as indicated in figure 1.*

If i *is benevolent, then* $b > 2$ *implies that* $a > 1$*, and* $b < 2$ *implies that* $a < 1$*; if* j *is benevolent, viceversa. In the former case,* (i_1, j_2) *is a strict Nash equilibrium; in the latter, on the other hand, the strategy in the profile are never best reply to each other.*

If, instead, they are malevolent, the profile (i_1, j_2) *is never a Nash equilibrium, but, alternatively,* $i_1 \in B_i(j_2)$ *oppure* $j_2 \in B_i(i_1)$*.*

Lastly, if they are both egoistic, all outcomes of the game are equivalent, and therefore all equilibria outcomes.

The next example considers a similar case, with the generalization of possible incomparability, hence without numerical payoffs.

Example 3

	j_1	j_2
i_1	α	β
i_2	γ	δ

(11)

Let outcomes in the second row and first column of table 11 be indifferent to each other for both player, as in table 10: therefore. for $k \in \{i, j\}$*,* $(\alpha, \gamma) \in Q_k$ *and* $(\gamma, \delta) \in Q_k$*. Strategic profile* γ *is thus again a Nash equilibrium.*

Unlike before, assume now that $(\alpha, \beta) \in U_j$*. By maternalism, the pair belongs also to* U_i*. Resulting relations between profiles* β *and* δ *may be constrained by requirements in the construction of the equivalence relation.*

For example, by equi-disposition, from $(\alpha, \beta) \in U_i$ *there follows that* $(\beta, \delta) \in U_i$*. Namely, if, instead,* $(\beta, \delta) \in G_i$*, then outcomes* α *and* δ*, which belong to the same block in* $\mathbf{Q}_i$ *would not share* *the better or the worse states; moreover,* $(\beta, \delta) \notin Q_i$*, because, otherwise, also* $(\alpha, \beta) \in Q_i$*, by transitivity of* Q_i*, against the assumption. There follows that* β *also is a Nash equilibrium, and its weakness structure ha the same form as* γ*'s, once indifference is replaced by uncomparability. (The two remaining strategic profiles are naturally also Nash equilibria.)*

(Profiles on the main diagonal are indifferent to each other for both players by transitivity, like in figure 1; profiles on the secondary diagonal are incomparable by transitivity of indifference and equi-disposition.)

On the issue of stability of the equilibria, the cases differ. If stability is intended as the possibility of being reached by rational individual substitution (as defined in section 2), that is, along rows and columns, and not diagonally, the equilibrium is example 2 is stable, and the equilibrium in example 3 is not.

It is a common feature of these examples, and of standard optimization in general, that a state which is indifferent to a maximal state, is itself maximal. There follows, for example, a well known instability of a non-strict Nash equilibrium, for a deviation from the equilibrium strategy x_i^*, providing "welfare" $q(x_i^*)$, to a non-equilibrium best reply x_i' is rational, given the indifference between (the states which, given the strategic circumstances, are the outcomes of) the two.

In the case of incomplete preferences, and of lack of equi-maximality, one the one hand, the instability may disappear, for (the maximal state which is the outcome of) the non-equilibrium best reply x_i' may not be indifferent to (the state which is the outcome of) the equilibrium strategy x_i^*, hence the deviation would not be rational. On the other hand, a different — and possibly stronger — instability may arise, because there may be (a state which is the outcome of) a strategy x_i' which is indifferent to (the state which is the outcome of) the equilibrium strategy x_i^*, and the deviation from x_i^* to x_i' be rational, yet the strategy x_i' may not be a best reply, because it may itself not be maximal. Strategy x_i' may be a rational "bridge" to a different strategy x_i'', with a higher "welfare" $q(x_i'')$.

The following example illustrates this case of a "double deviation".

Example 4

	j_1	j_2	j_3
i_1	α	β	ι
i_2	γ	δ	ϑ
i_3	ε	ζ	η

. (12)

Let γ, again, be a Nash equilibrium, and $\alpha \in Q_i(\gamma)$. But let $B_i(j_1) = \{i_2, i_3\}$, be i's best reply, with $(\varepsilon, \gamma) \in U_i$, and $(\varepsilon, \alpha) \in P_i$.

5 Generalization

In the context of two subjects, i add j, there are altogether $2^9 = 512$ possible axiomatizations to regiment i's intersubjective replies to j's kinds of preferences over pairs of states with a constant consumption of i's, if one does not separate between different strict preferences, i.e., between "malevolence" and "benevolence". These can be seen as all the possible ways to place numbers 1 or 0 in the cells of a 3×3 matrix of the kind of the tables 5 and 7 , were the convention may be that 0 denotes cases excluded, and 1 allowed for, by the axiomatization.

Out of these, $2^5 = 32$ axiomatizations agree in the treatment of the G_i and Q_i reply to G_j and Q_j, which generalizes the requirements of equation (1) to the case of non-regular preferences, like formulas (3), (4), and maternalism. These, again, are all the combinations of figures 1 or 0 in the cells of the same 3×3 matrix, but only in the third row and column, representing U_i and U_j, respectively. They are all composed in a possible Hasse diagram in figure 3, which shows the axiomatizations to be a graded partially ordered set with maximum and minimum elements. The segments denote relations of relative strength, increasing upwardly.

Finite combinatorics here guarantees axiomatization, but not all axiomatizations are equally "natural", either out of syntactical simplicity (as in the cases of (3) and (4)), or of the possibility of a paradigmatic interpretation (as in maternalism).

A notable and simple axiomatization is:

$$\Delta_i \cap (\cap_{k \neq i} Q_k) \cap G_i = \varnothing \tag{13}$$

this is the infimum of the graph, it is weaker than all others, and it allows for the maximum of possibilities.

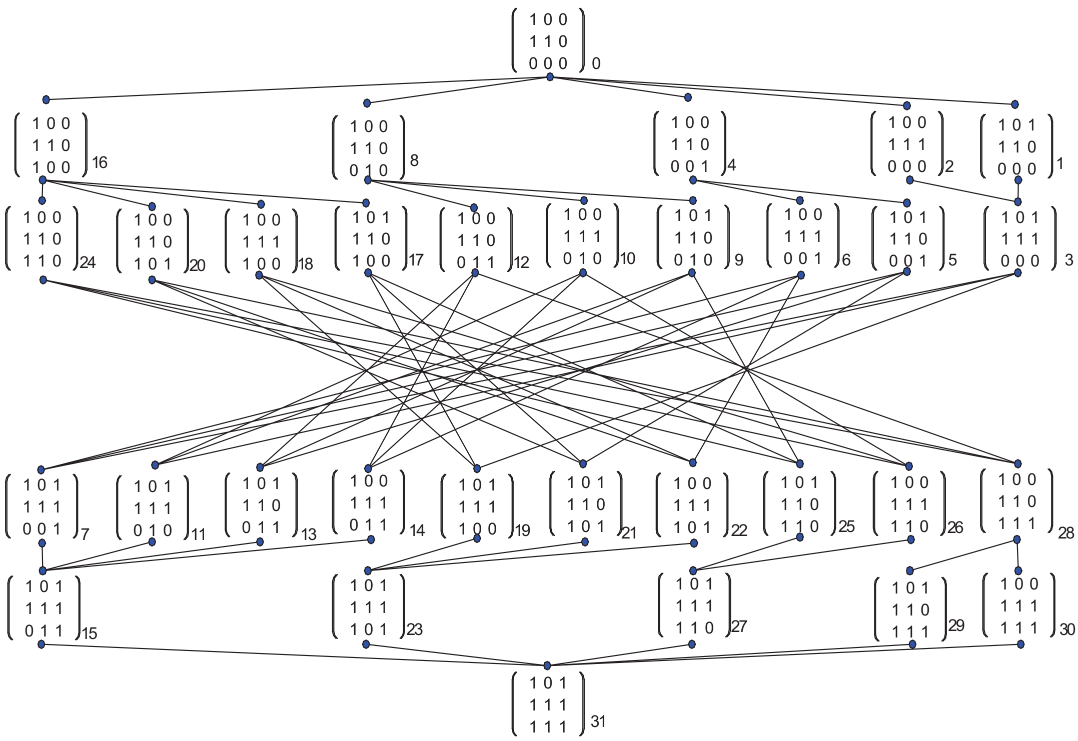

Figure 3: Hasse diagram of the 32 axiomatizations.

The axiomatizations are denoted by natural numbers, form **0** to **31**, which can be read out of the sequences of figures in the third row and column of each matrix, anticlockwise, to be interpreted as binary numbers. Hence, materialism is **28**, **30** is set inclusion (3), and **23** is set inclusion (4).

Whatever formal conditions one may consider, like syntactical simplicity, some sort of theory of judgment, like the one suggested concerning maternalism, seems to be a necessary guideline in the multitude of possible axiomatizations. Such theories will then undergo the tests of the semantic satisfaction of the axiomatizations they will select.

6 Comments and conclusions

It was the purpose of this note to examine the requirement of functional separability of the subjective utility function which formalizes intersubjectivity of individual preferences, and sensitivity to the welfare and preferences of others, as in [11], by the means of generalizations to the case of incomplete preferences. The main result is that the examined requirement breaks up into a number of axiomatically distinct, and variously related specifications. While the axiomatic requirement restricted to the case of regular preferences is of a standard and fairly straightforward interpretation, the resulting variety of the case of incomplete, hence non-regular, preferences seems to require an effort of conceptualization to provide for criteria for discrimination.

We believe the key turning point of the whole issue to remain the failure of numerical representability of incomplete preferences. This failure has the major consequence of the loss of a notion of an intrinsic "welfare" of each alternative state brought about by the utility function; indeed, in spite of the general indeterminacy of the non-uniqueness of such representations, the utility functions perform uniquely the important function of ascertaining the cases of "equal utility". Whenever two states are indifferent, any numerical representation is equally capable of exhibiting the "fact" that they *give the same utility* to the consumer.

According to this line of reasoning, then, this notion of intrinsic value of the states, whence there follows an intrinsic property of the relation between the states, i.e., equality of welfare, is lost because of the loss of a necessary relational property of the pairs, i.e., transitivity. That is to say, this approach contravenes established theoretical paradigms concerning the independence of intrinsic vs. extrinsic properties of entities (see, e.g., [6]). Because transitivity is only necessary, once this is re-establish by the means only of arbitrary restrictions of the residual relation, and possibly only by extrinsic criteria like equi-disposition in its various specification, it is hard to specify what kind of indifference, and what notion of intrinsic welfare value are regained. A relation of equivalence may well be obtained again, yet, lacking numerical representation, we are no longer in the position to state that two states are indifferent *because* they give the same "utility".

It is for this reason, we suggest, that it is no longer straightforward to interpret axiomatizations of intersubjectivity of preferences as forms of sensitivity to the *welfare* of others. When this made sense, with numerical representability, the requirement established that a subject sensitive *only* (i.e., given a constant own private consumption) to others' welfare, would not have had a strict preference between states all others deemed to be indifferent. But when the residual relation is partitioned, under whatever varying criteria, into a relation of indifference, and a polar different relation of incomparability, it becomes an open problem how the subject, whose own preferences are sensitivity to others', should reply to others' indifference, rather than to others' incomparability.

On the other hand, one may stipulate that, in line with the case of numerical representation, the subject should continue to receive "equal welfare" (given a constant own private consumption) from states all others deem to possess "equal welfare" (are indifferent). This criterion would implement a selection of the axiomatizations depicted in figure 3: maternalism would cease to be acceptable, for example, together with 17 other cases.

This argument, in fact, requires that a notion of welfare could still be available, in spite of numerical non-representability, i.e., the notion of a "non-quantitative" intrinsic welfare.[8] This notion would separate, in an obvious way, the case of indifference from the case of incomparability, for clearly the former, and not the latter, might be seen as a case of equal welfare. Hence, the subject's reply to others' incomparability might be discussed independently of the reply to others' indifference.

Example 5 *Axiomatization* **4** *is very stringent, and it combines what was just said on indifference, with a requirement on maternalism restricted to* U_j*, and a criterion of "no hand washing" on* G_j*.*

Consider U_j *alone. Then, requiring the third column to be everywhere 0 would mean to prevent* j *from being unable to compare pairs in* Δ_i*, and this does not seem to be any feature of sensitivity on behalf of* i*'s preferences with respect to* j*'s; and allowing* i*'s preferences to reply by* G_i *or* Q_i *would mean again to admit of a (paternalistic) evaluation on behalf of* i*, which is not justified by* j*'s lack of judgement.*

Finally, axiomatization **4***, is the "most diagonal" of all 32, and its symmetry may play a role in a game of reciprocity.*

On the other hand, if this "conceptual content" of indifference is considered, for some meth-

[8] Naturally, if a notion of "equal welfare" is admitted in the absence of numerical representation, a similar notion of "greater welfare", of x with respect to y whenever $(x, y) \in P_i$, should also be admitted in the same circumstances. In other words, the notion of welfare in general wouId turn out to be independent of numerical representation.

odological reasons, to be unavailable, and the distinctions between indifference and incomparability lies *only* in the transitivity (and it is not the case that indifference is transitive *because* the states yield equal welfare), then it seems to be more difficult to find reasons to discriminate between replies to the two possibilities, and, more in general, reasons for each of the two cases, only on the ground of relational properties of pairs: the reasons on the bases of which the subject is sensitive to others' evaluations of indifference, or to others' incomparability, will no longer depend on the (lack of) intrinsic evaluation, but on how the pairs do, or do not, "triangulate".

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