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Michele Gori

Working Paper N. 2014-04

DISEI, Università degli Studi di Firenze
Via delle Pandette 9, 50127 Firenze, Italia
www.disei.unifi.it

Selecting anonymous, neutral and reversal symmetric minimal majority rules*

Michele Gori
Dipartimento di Scienze per l'Economia e l'Impresa
Università degli Studi di Firenze
via delle Pandette 9, 50127, Firenze
e-mail: michele.gori@unifi.it

June 16, 2014

Abstract

Assuming that alternatives are three or more, we prove that if the set of anonymous, neutral and reversal symmetric minimal majority rules is nonempty, then it has at least two elements. We propose then further principles linked to equity and fairness that can be used to exclude some rules in that set and we show that, when alternatives are three, suitable combinations of those principles leads to identify a unique rule.

Keywords: Social welfare function; anonymity; neutrality; reversal symmetry; majority; linear order; group theory.

JEL classification: D71

1 Introduction

Consider a committee whose purpose is to provide a strict ranking of a given family of alternatives. Usually, committee members reveal their opinions on alternatives only after having chosen a procedure to get the final social outcome from their individual preferences. The determination of such a procedure, also called rule, is normally based on a preliminary agreement on the principles it should obey.

The well known principles of anonymity, neutrality, reversal symmetry and majority are often invoked as they are deemed able to guarantee a certain amount of equity and fairness in the collective decisions. The principle of anonymity states that identities of individuals are irrelevant to determine the social outcome; the principle of neutrality states that alternatives are equally treated; the principle of reversal symmetry states that a complete change in each committee member's mind about her own ranking of alternatives implies a complete change in the social outcome; the principle of majority states that if the number of people preferring an alternative to another one is greater than or equal to a fixed majority threshold, then the former alternative has to be socially preferred to the latter one. As the majority principle, for a given majority threshold, may be inconsistent with any social outcome because of the presence of Condorcet-cycles, Bubboloni and Gori (2013) recently introduced a new version of that principle, called minimal majority principle: it requires

*The author greatly thanks Daniela Bubboloni for suggesting some of the issues considered in the paper and for providing useful comments which allowed to improve and clarify many arguments. The author is also grateful to Pierluigi Zezza for computing via a CAS the representatives of the orbits in some special cases: such an empirical material strongly helped guide the research and allowed to prove Statement 1 of Proposition 13 when individuals are five and alternatives are four.

that the social outcome has to be consistent with all the majority thresholds which do not generate Condorcet-cycles.

Under the assumption that individual preferences are expressed in the form of strict rankings, Bubboloni and Gori (2014, Corollary 16) prove that the set of anonymous, neutral and reversal symmetric minimal majority rules, denoted by $\mathcal{F}^G_{\min}$, is nonempty if and only if $\gcd(h, n!) = 1$, where h is the number of committee members and n is the number of alternatives to be ranked[1]. Moreover, under the assumption $\gcd(h, n!) = 1$, if alternatives are two, then the simple majority is the unique element in $\mathcal{F}^G_{\min}$, while if alternatives are three or more, then $\mathcal{F}^G_{\min}$ has at least two elements[2]. That means that, when alternatives are at least three, founding an agreement on the principles of anonymity, neutrality, reversal symmetry and minimal majority is not enough to select a unique rule. Thus, further shared principles are needed to finally decide which rule to employ for the collective decision. The main purpose of the present paper is exactly to discuss some principles that committee members may use to exclude some rules in $\mathcal{F}^G_{\min}$ and that, in our opinion, are still linked to the intuitive concepts of equity and fairness. As we are going to show, those principle sometimes lead to make only one rule survive.

Let us introduce them. Given a preference profile p (that is, a list of individual preferences expressed as strict rankings) and two strict rankings q_1 and q_2 (interpreted as possible social outcomes), we say that q_1 gets more votes than q_2 (according to p) if the number of individuals whose preferences are equal to q_2 is greater than the number of individuals whose preferences are equal to q_1. We say that q_1 is Pareto superior to q_2 (according to p) if, for every pair of alternatives, individuals who agree with q_1 with respect to the two alternatives are at least as many as individuals who agree with q_2 and, for a particular pair of alternatives, they are more. Consider now a set $\mathcal{G}$ of rules. We say that a rule $F \in \mathcal{G}$ satisfies the *most votes principle* in $\mathcal{G}$ if, for every rule $F' \in \mathcal{G}$ and every preference profile p, the social outcome associated with p by F' does not get more votes than the social outcome associated with p by F. Analogously, we say that a rule $F \in \mathcal{G}$ satisfies the *Pareto principle* in $\mathcal{G}$ if, for every rule $F' \in \mathcal{G}$ and every preference profile p, the social outcome associated with p by F' is not Pareto superior to the social outcome associated with p by F. We denote by $M(\mathcal{G})$ the subset of rules of $\mathcal{G}$ satisfying the most votes principle in $\mathcal{G}$, and by $P(\mathcal{G})$ the subset of rules of $\mathcal{G}$ satisfying the Pareto principle in $\mathcal{G}$.

Assuming that committee members agree to pick their aggregation rule in the set $\mathcal{F}^G_{\min}$, we think that rules in $M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})$ should be preferred. Indeed, in our opinion, their further properties assure yet more equitable and fairer collective choices. Unfortunately, it is not true that such a set is nonempty for all h and n.

Focusing at first on the special case where alternatives are three and under the assumption $\gcd(h, n!) = 1$, we prove that $M(\mathcal{F}^G_{\min})$ and $P(\mathcal{F}^G_{\min})$ are always nonempty. Moreover, we show that if $h \in \{5, 7, 11, 13\}$, then $M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})$ is a singleton, while if $h \notin \{5, 7, 11, 13\}$, then that set is empty. In the latter case, we have then that demanding a rule satisfying both the most votes principle and the Pareto principle in $\mathcal{F}^G_{\min}$ is too restrictive. A possible way out can be to find a preliminary agreement on which is the most compelling principle between the two. If committee members decide that it is the most votes principle, then they have to first look at the set $M(\mathcal{F}^G_{\min})$ and, if necessary, at the set $P(M(\mathcal{F}^G_{\min}))$. If committee members decide instead that the most compelling is the Pareto principle, then they first have to consider $P(\mathcal{F}^G_{\min})$ and later, if necessary, $M(P(\mathcal{F}^G_{\min}))$. According to this idea we prove that, while $P(\mathcal{F}^G_{\min})$ is never a singleton, $M(\mathcal{F}^G_{\min})$ is a singleton if and only if $h \in \{5, 7, 11\}$. That means that, differently from the Pareto principle, the refinement determined by the most votes principle alone sometimes selects a unique rule. We also prove that $P(M(\mathcal{F}^G_{\min}))$ and $M(P(\mathcal{F}^G_{\min}))$ are always singletons and that if $h \in \{5, 7, 11, 13\}$, then they are both equal to $M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})$, while if $h \notin \{5, 7, 11, 13\}$, then they are not equal. As a consequence, if $h \in \{5, 7, 11, 13\}$, we have that any combination of the most votes and Pareto principles leads to the same unique outcome, while if $h \notin \{5, 7, 11, 13\}$, the order the principles are applied is not inessential.

[1]Note that Moulin (1983, Theorem 1, p.23) first understood the importance of condition $\gcd(h, n!) = 1$, proving that it is a necessary and sufficient condition for the existence of anonymous, neutral social choice functions satisfying the unanimity condition.

[2]The first fact is well known, while the second one is not. However, it is an immediate consequence of Theorems 1 and 2 so that we do not directly prove it in the paper.

We finally show that, for every value of h, both the unique element in $P(M(\mathcal{F}^G_{\min}))$ and the one in $M(P(\mathcal{F}^G_{\min}))$ can be fully described via a simple algorithm whose pseudo-code is proposed in Section 8. We stress that the possibility to design those algorithms is strongly related to the constructive way to prove the mentioned results, which in turn is based on the algebraic techniques introduced in Bubboloni and Gori (2013, 2014). In our opinion, such a constructive approach to proofs and its by-products (counting the rules, finding algorithms) deserve to be carefully deepened particularly in respect of the goals and methods of computational social choice (see Chevaleyre et al., 2007).

When the alternatives are at least four and $\gcd(h, n!) = 1$, we show instead that $M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})$ is empty, while $P(M(\mathcal{F}^G_{\min}))$ and $M(P(\mathcal{F}^G_{\min}))$ have more than one element. That shows in particular that, with more than three alternatives, no combination of the most votes and Pareto principles is able to select a unique anonymous, neutral and reversal symmetric minimal majority rule, so that the analysis of further or different principles is necessary.

Of course, even though in the paper we focus only on the most votes and Pareto principles, our approach for selecting rules in $\mathcal{F}^G_{\min}$ can be employed to other principles. Indeed, for any conceivable set of reasonable principles, one may ask how many rules are selected by a given combination of them, whether there is a special combination selecting exactly one rule, and, when uniqueness is got, whether there is a simple and efficient algorithm for the computation of the social outcome. At the same time, we strongly believe that in a number of different situations the algebraic machinery developed in Bubboloni and Gori (2013, 2014), which proves to be fundamental to manage the issues considered in this paper, can be successfully applied.

2 Definitions and main results

From now on let $h, n \in \mathbb{N}$ with $h, n \geq 2$ be fixed. Let $H = \{1, \ldots, h\}$ be the set of individuals and $N = \{1, \ldots, n\}$ be the set of alternatives. A *preference* on N is a linear order on N, that is, a complete, transitive and antisymmetric binary relation on N. We denote by $\mathcal{L}(N)$ the set of preferences on N. Given $p_0 \in \mathcal{L}(N)$ and $x, y \in N$, if $(x, y) \in p_0$ and $(y, x) \notin p_0$, then we say that x is *preferred* to y according to p_0 and we sometimes write $x >_{p_0} y$. We identify preferences on N with column vectors: for instance, the vector $[2, 1, 3]^T$ represents the preference on $\{1, 2, 3\}$ according to which 2 is preferred to 1 and 3, and 1 is preferred to 3. A *preference profile* is an element of $\mathcal{P} = \mathcal{L}(N)^h$. If $p \in \mathcal{P}$ and $i \in H$, the i-th component of p is denoted by p_i and represents the preference of individual i. Any $p \in \mathcal{P}$ can be identified with the $(n \times h)$-matrix whose i-th column is the column vector representing p_i. A *rule* (or social welfare function) is a function from $\mathcal{P}$ to $\mathcal{L}(N)$. The set of rules is denoted by $\mathcal{F}$.

Let S_h be the set of bijective functions from H into H, S_n be the set of bijective functions from N into N, and Ω be the subset of S_n whose elements are the identity function and the reversal map ρ_0, defined, for every $r \in N$, as $\rho_0(r) = n - r + 1$. Those sets, whose elements are called permutations, are groups with the product given by the right-to-left composition[3] and neutral element given by the identity function id. Given now $[a_1, \ldots, a_n]^T \in \mathcal{L}(N)$ and $\psi \in S_n$, define

$$\psi[a_1, \ldots, a_n]^T = [\psi(a_1), \ldots, \psi(a_n)]^T,$$

$$[a_1, \ldots, a_n]^T id = [a_1, \ldots, a_n]^T,$$

$$[a_1, \ldots, a_n]^T \rho_0 = [a_{\rho_0(1)}, \ldots, a_{\rho_0(n)}]^T = [a_n, \ldots, a_1]^T.$$

Let us consider the group $G = S_h \times S_n \times \Omega$, and define, for every $p \in \mathcal{P}$ and $(\varphi, \psi, \rho) \in G$, $p^{(\varphi,\psi,\rho)} \in \mathcal{P}$ as the preference profile such that, for every $i \in H$,

$$\left(p^{(\varphi,\psi,\rho)}\right)_i = \psi p_{\varphi^{-1}(i)} \rho.$$

[3]Let $k \in \mathbb{N}$ and $f_1, f_2 \in S_k$. Then $f_1 f_2 \in S_k$ is the function such that, for every $x \in \{1, \ldots, k\}$, $f_1 f_2(x) = f_1(f_2(x))$. Note that in group theory is more frequent the left-to-right notation. Any notation and basic results for permutations groups used in the paper are standard (see, for instance, Wielandt (1964) and Rose (1978)).

The preference profile $p^{(\varphi,\psi,\rho)}$ is then obtained by p according to the following rules: for every $i \in H$, individual i is renamed $\varphi(i)$; for every $x \in N$, alternative x is renamed $\psi(x)$; for every $r \in N$, alternatives whose rank is r are moved to rank $\rho(r)$. For instance, if $n = 3$, $h = 5$ and

$$p = \begin{bmatrix} 3 & 1 & 2 & 3 & 2 \\ 2 & 2 & 1 & 2 & 3 \\ 1 & 3 & 3 & 1 & 1 \end{bmatrix}, \quad \varphi = (134)(25), \quad \psi = (12), \quad \rho = \rho_0 = (13),$$

then we have

$$p^{(\varphi,\psi,\rho_0)} = \begin{bmatrix} 2 & 2 & 2 & 3 & 3 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 1 & 3 & 1 & 2 \end{bmatrix}.$$

Later on, we will write the i-th component of $p^{(\varphi,\psi,\rho)}$ simply as $p_i^{(\varphi,\psi,\rho)}$.

A rule F is said anonymous, neutral and reversal symmetric, or briefly G-*symmetric*, if, for every $p \in \mathcal{P}$ and $(\varphi, \psi, \rho) \in G$,

$$F(p^{(\varphi,\psi,\rho)}) = \psi F(p)\rho.$$

The set of G-symmetric rules is denoted by $\mathcal{F}^G$.

Given $\nu \in \mathbb{N} \cap (h/2, h]$, we define, for every $p \in \mathcal{P}$, the set

$$C_\nu(p) = \Big\{ q_0 \in \mathcal{L}(N) : \forall x, y \in N, |\{i \in H : x >_{p_i} y\}| \geq \nu \;\; \Rightarrow \;\; x >_{q_0} y \Big\},$$

that is, the set of preferences having x preferred to y whenever, according to the preference profile p, at least ν individuals prefer x to y. Note that if $\nu, \nu' \in \mathbb{N} \cap (h/2, h]$ and $\nu \leq \nu'$, then we have that, for every $p \in \mathcal{P}$, $C_\nu(p) \subseteq C_{\nu'}(p)$. It is known that[4] $C_\nu(p) \neq \varnothing$ for all $p \in \mathcal{P}$ if and only if $\nu > \frac{n-1}{n}h$. For every $p \in \mathcal{P}$, define also

$$\nu(p) = \min\{\nu \in \mathbb{N} \cap (h/2, h] : C_\nu(p) \neq \varnothing\},$$

and observe that the definition is well posed as $C_h(p) \neq \varnothing$. A rule F is said a *minimal majority rule* if, for every $p \in \mathcal{P}$, $F(p) \in C_{\nu(p)}(p)$. The set of minimal majority rules, denoted by $\mathcal{F}_{\min}$, is clearly nonempty.

Consider now the set of G-symmetric minimal majority rules, that is, the set $\mathcal{F}^G_{\min} = \mathcal{F}^G \cap \mathcal{F}_{\min}$. As proved in Bubboloni and Gori (2014, Corollary 16 and Section 7.2), we have that

$$\mathcal{F}^G_{\min} \neq \varnothing \quad \text{if and only if} \quad \gcd(h, n!) = 1. \tag{1}$$

Moreover, under the assumption $\gcd(h, n!) = 1$, it is immediate to prove that $n = 2$ implies that the simple majority rule is the unique element in $\mathcal{F}^G_{\min}$, while $n \geq 3$ implies $|\mathcal{F}^G_{\min}| \geq 2$ (as follows by Theorems 1 and 2 below). That means that, when at least three alternatives are considered, if the principles of anonymity, neutrality, reversal symmetry and minimal majority are not contradictory, then they are consistent with two or more rules. In what follows, we propose further reasonable principles that can be used to select rules in $\mathcal{F}^G_{\min}$ when $\gcd(h, n!) = 1$. Those principles have been discussed in the introduction: here we propose their formalization within our framework. As we will show, in some cases those principles allow to identify a unique rule.

Given a subset $\mathcal{G}$ of $\mathcal{F}$, let us define

$$M(\mathcal{G}) = \Big\{ F \in \mathcal{G} : \forall F' \in \mathcal{G}, \; \forall p \in \mathcal{P}, |\{i \in H : p_i = F(p)\}| \geq |\{i \in H : p_i = F'(p)\}| \Big\}.$$

Thus, $F \notin M(\mathcal{G})$ means there are $F' \in \mathcal{G}$ and $p \in \mathcal{P}$ such that people expressing $F'(p)$ are more than the ones expressing $F(p)$. We say that rules in $M(\mathcal{G})$ satisfy the most votes principle in $\mathcal{G}$. Note that $M(M(\mathcal{G})) = M(\mathcal{G})$ and that if $|\mathcal{G}| \leq 1$, then $M(\mathcal{G}) = \mathcal{G}$.

[4] See, for instance, Theorem 10 in Bubboloni and Gori (2013).

For every $p \in \mathcal{P}$, $q_0 \in \mathcal{L}(N)$ and $x, y \in N$ with $x \neq y$, let us denote by $A(p, q_0, x, y)$ the number of individuals who, according to p, express the same opinion as q_0 on alternatives x and y, that is,

$$A(p, q_0, x, y) = \begin{cases} |\{i \in H : (x, y) \in p_i\}| & \text{if } (x, y) \in q_0, \\ |\{i \in H : (y, x) \in p_i\}| & \text{if } (y, x) \in q_0. \end{cases}$$

Of course, $A(p, q_0, x, y) \in \{0, \dots, h\}$ and $A(p, q_0, x, y) = A(p, q_0, y, x)$. Let us consider the set

$$\mathcal{C} = \{(x, y) \in N^2 : x < y\}$$

whose order is $\frac{n(n-1)}{2}$, and, for every $p \in \mathcal{P}$ and $q_0 \in \mathcal{L}(N)$, define the vector

$$A(p, q_0) = \big(A(p, q_0, x, y)\big)_{(x,y)\in\mathcal{C}} \in \{0, \dots, h\}^{\frac{n(n-1)}{2}}.$$

Given now a subset $\mathcal{G}$ of $\mathcal{F}$, let us define[5]

$$P(\mathcal{G}) = \Big\{F \in \mathcal{G} : \forall F' \in \mathcal{G},\ \forall p \in \mathcal{P}, A(p, F'(p)) \not> A(p, F(p))\Big\}.$$

We say that rules in $P(\mathcal{G})$ satisfy the Pareto principle in $\mathcal{G}$. Note that $P(P(\mathcal{G})) = P(\mathcal{G})$ and that if $|\mathcal{G}| \leq 1$, then $P(\mathcal{G}) = \mathcal{G}$.

The following theorems describe some properties of the sets $M(\mathcal{F}^G_{\min})$, $P(\mathcal{F}^G_{\min})$, $M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})$, $P(M(\mathcal{F}^G_{\min}))$ and $M(P(\mathcal{F}^G_{\min}))$. We recall that $\gcd(h, n!) \geq 2$ implies that all those sets are empty since $\mathcal{F}^G_{\min}$ is empty. Moreover, when $n = 2$ and $\gcd(h, n!) = 1$, they all have a unique element given by the simple majority rule.

Theorem 1. *Let $n = 3$ and $gcd(h, n!) = 1$.*

1. *If $h \in \{5, 7, 11\}$, then $|M(\mathcal{F}^G_{\min})| = 1$.*
2. *If $h \notin \{5, 7, 11\}$, then $|M(\mathcal{F}^G_{\min})| \geq 2$.*
3. *$|P(\mathcal{F}^G_{\min})| \geq 2$.*
4. *If $h \in \{5, 7, 11, 13\}$, then $|M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})| = 1$.*
5. *If $h \notin \{5, 7, 11, 13\}$, then $M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min}) = \varnothing$.*
6. *$|P(M(\mathcal{F}^G_{\min}))| = 1$.*
7. *$|M(P(\mathcal{F}^G_{\min}))| = 1$.*
8. *If $h \in \{5, 7, 11, 13\}$, $M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min}) = P(M(\mathcal{F}^G_{\min})) = M(P(\mathcal{F}^G_{\min}))$.*
9. *If $h \notin \{5, 7, 11, 13\}$, $P(M(\mathcal{F}^G_{\min})) \neq M(P(\mathcal{F}^G_{\min}))$.*

Theorem 1 states in particular that if $n = 3$ and $\gcd(h, n!) = 1$, then both $P(M(\mathcal{F}^G_{\min}))$ and $M(P(\mathcal{F}^G_{\min}))$ are singletons. Moreover, denoting by F^{MP} the unique element in $P(M(\mathcal{F}^G_{\min}))$ and by F^{PM} the unique element in $M(P(\mathcal{F}^G_{\min}))$, we have that $F^{MP} = F^{PM}$ if and only if $h \in \{5, 7, 11, 13\}$. We finally emphasize that the strategy used to prove Theorem 1 allows to determine simple algorithms to compute the value of F^{MP} and F^{PM} on every preference profile. Those algorithms are described in Section 8.

The next theorem shows instead that, when the alternatives are at least four, any combination of the most votes and Pareto principles is unable to select a unique rule.

Theorem 2. *Let $n \geq 4$ and $gcd(h, n!) = 1$.*

1. *$M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min}) = \varnothing$.*
2. *$|P(M(\mathcal{F}^G_{\min}))| \geq 2$.*
3. *$|M(P(\mathcal{F}^G_{\min}))| \geq 2$.*

[5] Given $k \in \mathbb{N}$ and $v = (v_i)_{i=1}^k, w = (w_i)_{i=1}^k \in \mathbb{R}^k$, we write $v \geq w$ when $v_i \geq w_i$ for all $i \in \{1, \dots, k\}$, and we write $v > w$ when $v \geq w$ and $v \neq w$. We also use the symbol $\not>$ with the obvious meaning.

3 Action on the set of preference profile

As proved in Bubboloni and Gori (2014, Proposition 1), for every $p \in \mathcal{P}$ and $(\varphi_1, \psi_1, \rho_1), (\varphi_2, \psi_2, \rho_2) \in G$, we have

$$p^{(\varphi_1\varphi_2, \psi_1\psi_2, \rho_1\rho_2)} = \left(p^{(\varphi_2, \psi_2, \rho_2)}\right)^{(\varphi_1, \psi_1, \rho_1)}.$$

Then, the function[6] $f : G \to \mathrm{Sym}(\mathcal{P})$ defined, for every $(\varphi, \psi, \rho) \in U$, as

$$f(\varphi, \psi, \rho) : \mathcal{P} \to \mathcal{P}, \quad p \mapsto p^{(\varphi,\psi,\rho)},$$

is well posed and it is an action of the group G on the set $\mathcal{P}$. For every $p \in \mathcal{P}$, we define the orbit of p as $p^G = \{p^{(\varphi,\psi,\rho)} \in \mathcal{P} : (\varphi, \psi, \rho) \in G\}$, and the stabilizer of p as

$$\mathrm{Stab}_G(p) = \left\{(\varphi, \psi, \rho) \in G : p^{(\varphi,\psi,\rho)} = p\right\}.$$

The set of orbits $\mathcal{O} = \{p^G : p \in \mathcal{P}\}$ is a partition[7] of $\mathcal{P}$ and we put $|\mathcal{O}| = R$. Any vector $(p^j)_{j=1}^R \in \mathcal{P}^R$ such that $\{p^{j\,G} : j \in \{1, \ldots, R\}\} = \mathcal{O}$, is called a system of representatives of the orbits. The set of the systems of representatives of the orbits is nonempty and denoted by $\mathfrak{S}$.

4 Preliminary results

For every $p \in \mathcal{P}$, let us define the following sets:

$$S_1^G(p) = \left\{q_0 \in \mathcal{L}(N) : \forall (\varphi, \psi, \rho) \in \mathrm{Stab}_G(p),\ \psi q_0 \rho = q_0\right\},$$

$$S_2^G(p) = S_1^G(p) \cap C_{\nu(p)}(p),$$

$$S_M^G(p) = \left\{q_0 \in S_2^G(p) : \forall q_1 \in S_2^G(p), |\{i \in H : p_i = q_0\}| \geq |\{i \in H : p_i = q_1\}|\right\},$$

$$S_P^G(p) = \left\{q_0 \in S_2^G(p) : \forall q_1 \in S_2^G(p),\ A(p, q_1) \not> A(p, q_0)\right\},$$

$$S_{MP}^G(p) = \left\{q_0 \in S_M^G(p) : \forall q_1 \in S_M^G(p),\ A(p, q_1) \not> A(p, q_0)\right\},$$

$$S_{PM}^G(p) = \left\{q_0 \in S_P^G(p) : \forall q_1 \in S_P^G(p), |\{i \in H : p_i = q_0\}| \geq |\{i \in H : p_i = q_1\}|\right\}.$$

Proposition 3. *Let $p \in \mathcal{P}$, $q_0 \in \mathcal{L}(N)$, $(\varphi, \psi, \rho) \in G$ and $x, y \in N$ with $x \neq y$. Then*

$$A(p^{(\varphi,\psi,\rho)}, \psi q_0 \rho, x, y) = A(p, q_0, \psi^{-1}(x), \psi^{-1}(y)).$$

Proof. Assume first that $(x, y) \in \psi q_0 \rho$ and $\rho = \rho_0$. Observe first that $(x, y) \in \psi q_0 \rho$ if and only if $(\psi^{-1}(x), \psi^{-1}(y)) \in q_0 \rho$ if and only if $(\psi^{-1}(y), \psi^{-1}(x)) \in q_0$. Then we have to check that

$$|\{i \in H : (x, y) \in p_i^{(\varphi,\psi,\rho)}\}| = |\{i \in H : (\psi^{-1}(y), \psi^{-1}(x)) \in p_i\}|.$$

But that equality holds true as

$$|\{i \in H : (x, y) \in p_i^{(\varphi,\psi,\rho))}\}| = |\{i \in H : (x, y) \in \psi p_{\varphi^{-1}(i)} \rho\}| = |\{i \in H : (x, y) \in \psi p_i \rho\}| =$$

$$|\{i \in H : (\psi^{-1}(x), \psi^{-1}(y)) \in p_i \rho\}| = |\{i \in H : (\psi^{-1}(y), \psi^{-1}(x)) \in p_i\}|.$$

All the other cases can be analogously analised. □

[6] $\mathrm{Sym}(\mathcal{P})$ is the set of bijective functions from $\mathcal{P}$ to $\mathcal{P}$. It is a group with respect to the right-to-left composition.

[7] In the paper, a partition of a nonempty set X is a family of nonempty pairwise disjoint subsets of X whose union is X.

Proposition 4. *Let $p \in \mathcal{P}$, $(\varphi, \psi, \rho) \in G$ and $j \in \{1, 2, M, P, MP, PM\}$. Then*

$$S_j^G(p^{(\varphi,\psi,\rho)}) = \psi S_j^G(p)\rho.$$

Proof. It is well known that

$$\mathrm{Stab}_G(p^{(\varphi,\psi,\rho)}) = (\varphi, \psi, \rho)\,\mathrm{Stab}_G(p)\,(\varphi^{-1}, \psi^{-1}, \rho^{-1}).$$

Moreover, as proved in Bubboloni and Gori (2014, Lemma 7), we have that

$$C_{\nu(p^{(\varphi,\psi,\rho)})}(p^{(\varphi,\psi,\rho)}) = \psi C_{\nu(p)}(p)\rho.$$

Using the above equalities, Proposition 3 and the fact that Ω is abelian, a routine computation allows to complete the proof. $\square$

Proposition 5. *Let* $\gcd(h, n!) \neq 1$. *Then there exists $p \in \mathcal{P}$ such that $S_1^G(p) = \varnothing$.*

Proof. By the proof of Theorem 5 in Bubboloni and Gori (2013), there exists $p \in \mathcal{P}$ and an element $(\varphi, \psi, id) \in \mathrm{Stab}_G(p)$ with $\psi \neq id$. Assume that $q_0 \in S_1^G(p)$: then q_0 should satisfy $\psi q_0 = q_0$, that is, $\psi = id$, a contradiction. $\square$

Proposition 6. *Let* $\gcd(h, n!) = 1$. *Then, for every $p \in \mathcal{P}$, $S_2^G(p) \neq \varnothing$.*

Proof. See the proof of Theorem 10 in Bubboloni and Gori (2014), recalling that $\gcd(h, n!) = 1$ implies that G is a regular group. $\square$

Proposition 7. *Let* $\gcd(h, n!) = 1$. *Then, for every $p \in \mathcal{P}$, $S_M^G(p) \neq \varnothing$.*

Proof. Given $p \in \mathcal{P}$, by Proposition 6 we have that $S_2^G(p) \neq \varnothing$. As a consequence, the set $\big\{|\{i \in H : p_i = q_0\}| : q_0 \in S_2^G(p)\big\} \subseteq \mathbb{N}_0$ is nonempty and finite and thus has a maximum m. Consider then any $q_0^* \in S_2^G(p)$ such that $|\{i \in H : p_i = q_0^*\}| = m$. Then $q_0^* \in S_M^G(p) \neq \varnothing$. $\square$

Proposition 8. *Let* $\gcd(h, n!) = 1$. *Then, for every $p \in \mathcal{P}$, $S_P^G(p) \neq \varnothing$.*

Proof. Fix $p \in \mathcal{P}$ and let $\succeq$ be the relation on the nonempty set $S_2^G(p)$ defined as follows: for every $q_0, q_1 \in S_2^G(p)$, we set $q_0 \succeq q_1$ if, for every $x, y \in N$ with $x \neq y$, $A(p, q_0, x, y) \geq A(p, q_1, x, y)$.

It is immediate to check that $\succeq$ is reflexive and transitive. Let us prove now that $\succeq$ is also antisymmetric. Consider $q_0, q_1 \in S_2^G(p)$ and assume that $q_0 \succeq q_1$ and $q_1 \succeq q_0$. Then, for every $x, y \in N$ with $x \neq y$,

$$A(p, q_0, x, y) \geq A(p, q_1, x, y), \quad \text{and} \quad A(p, q_1, x, y) \geq A(p, q_0, x, y).$$

that is, $A(p, q_0, x, y) = A(p, q_1, x, y)$. Assume now by contradiction that $q_0 \neq q_1$. Then there exist $x', y' \in N$ such that $(x', y') \in q_0$ and $(y', x') \in q_1$. Thus

$$A(p, q_0, x', y') = |\{i \in H : (x', y') \in p_i\}| \quad \text{and} \quad A(p, q_1, x', y') = |\{i \in H : (y', x') \in p_i\}|.$$

Since $|\{i \in H : (x', y') \in p_i\}| = |\{i \in H : (y', x') \in p_i\}|$ and

$$|\{i \in H : (x', y') \in p_i\}| + |\{i \in H : (y', x') \in p_i\}| = h,$$

we have that $2 \mid h$ and the contradiction is found. Since $\succeq$ is reflexive, antisymmetric and transitive and $S_2^G(p)$ is finite, the set of maximal elements of $\succeq$ is nonempty and equal to $S_P^G(p)$. $\square$

Proposition 9. *Let* $\gcd(h, n!) = 1$. *Then, for every $p \in \mathcal{P}$, $S_{MP}^G(p) \neq \varnothing$.*

Proof. Repeat the proof of Proposition 8, using $S_M^G(p) \neq \varnothing$ guaranteed by Proposition 7. $\square$

Proposition 10. *Let* $\gcd(h, n!) = 1$. *Then, for every $p \in \mathcal{P}$, $S_{PM}^G(p) \neq \varnothing$.*

Proof. Repeat the proof of Proposition 7, using $S_P^G(p) \neq \varnothing$ guaranteed by Proposition 8. $\square$

Proposition 11. *Let* $\gcd(h, n!) = 1$*. Then, for every* $p \in \mathcal{P}$*,*

$$|C_{\nu(p)}(p)| = 1 \quad \Rightarrow \quad C_{\nu(p)}(p) = S_2^G(p) = S_M^G(p) = S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p).$$

Proof. For every $p \in \mathcal{P}$, $C_{\nu(p)}(p) \supseteq S_2^G(p) \supseteq S_M^G(p) \supseteq S_{MP}^G(p)$ and $C_{\nu(p)}(p) \supseteq S_2^G(p) \supseteq S_P^G(p) \supseteq S_{PM}^G(p)$. Moreover, by Propositions 6, 7, 8, 9, and 10, the sets $S_2^G(p)$, $S_M^G(p)$, $S_P^G(p)$, $S_{MP}^G(p)$, and $S_{PM}^G(p)$ are nonempty. That completes the proof. $\square$

The next crucial propositions are proved in Sections 6 and 7.

Proposition 12. *Let* $n = 3$ *and* $\gcd(h, n!) = 1$*.*

1. *If* $h \in \{5, 7, 11\}$*, then* $|S_M^G(p)| = 1$ *for all* $p \in \mathcal{P}$*.*
2. *If* $h \notin \{5, 7, 11\}$*, then there exists* $p \in \mathcal{P}$ *such that* $|S_M^G(p)| \geq 2$*.*
3. *There exists* $p \in \mathcal{P}$ *such that* $|S_P^G(p)| \geq 2$*.*
4. $|S_{MP}^G(p)| = 1$ *for all* $p \in \mathcal{P}$*.*
5. $|S_{PM}^G(p)| = 1$ *for all* $p \in \mathcal{P}$*.*
6. *If* $h \in \{5, 7, 11, 13\}$*, then* $S_{MP}^G(p) = S_{PM}^G(p) = S_M^G(p) \cap S_P^G(p)$ *for all* $p \in \mathcal{P}$*.*
7. *If* $h \notin \{5, 7, 11, 13\}$*, then there exists* $p \in \mathcal{P}$ *such that* $S_M^G(p) \cap S_P^G(p) = \varnothing$*. In particular,* $S_{MP}^G(p) \neq S_{PM}^G(p)$*.*

Proposition 13. *Let* $n \geq 4$ *and* $\gcd(h, n!) = 1$*.*

1. *There exists* $p \in \mathcal{P}$ *such that* $S_M^G(p) \cap S_P^G(p) = \varnothing$*.*
2. *There exists* $p \in \mathcal{P}$ *such that* $|S_M^G(p) \cap S_P^G(p)| \geq 2$*,* $|S_{MP}^G(p)| \geq 2$*, and* $|S_{MP}^G(p)| \geq 2$*.*

5 Proofs of Theorems 1 and 2

To begin with, we state some results proved in Bubboloni and Gori (2014).

- If $F \in \mathcal{F}^G$, then $F(p) \in S_1^G(p)$ for all $p \in \mathcal{P}$.
- If $F \in \mathcal{F}^G_{\min}$, then $F(p) \in S_2^G(p)$ for all $p \in \mathcal{P}$.
- Fix $(p^j)_{j=1}^R \in \mathfrak{S}$. For every $(q_j)_{j=1}^R \in \times_{j=1}^R S_1^G(p^j)$, there exists a unique element in $\mathcal{F}^G$, denoted by $\Psi\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$, mapping p^j into q_j for all $j \in \{1, \ldots, R\}$. Moreover, the function

$$f : \times_{j=1}^R S_1^G(p^j) \to \mathcal{F}^G, \quad (q_j)_{j=1}^R \mapsto f\left((q_j)_{j=1}^R\right) = \Psi\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right] \tag{2}$$

  is bijective and we have

$$f\left(\times_{j=1}^R S_2^G(p^j)\right) = \mathcal{F}^G_{\min}, \quad |\mathcal{F}^G_{\min}| = \prod_{j=1}^R |S_2^G(p^j)|. \tag{3}$$

- $\mathcal{F}^G \neq \varnothing$ if and only if $\mathcal{F}^G_{\min} \neq \varnothing$ if and only if $\gcd(h, n!) = 1$.

Proposition 14. *Let* $F \in M(\mathcal{F}^G_{\min})$*. Then, for every* $p \in \mathcal{P}$*,* $F(p) \in S_M^G(p)$*.*

Proof. Since in particular $F \in \mathcal{F}^G_{\min}$, we know that, for every $p \in \mathcal{P}$, $F(p) \in S^G_2(p)$. Assume by contradiction there exists $p^* \in \mathcal{P}$ such that $F(p^*) \notin S^G_M(p^*)$. Then there exists $q^*_0 \in S^G_2(p^*)$ such that $|\{i \in H : p^*_i = F(p^*)\}| < |\{i \in H : p^*_i = q^*_0\}|$. Consider now any $(p^j)^R_{j=1} \in \mathfrak{S}$ such that $p^1 = p^*$, and define $(q_j)^R_{j=1} \in \times^R_{j=1} S^G_2(p^j)$ as $q_1 = q^*_0$ and, for every $j \in \{2, \dots, R\}$, $q_j = F(p^j)$. Then the rule $F' = \Psi[(p^j)^R_{j=1}, (q_j)^R_{j=1}] \in \mathcal{F}^G_{\min}$ is such that

$$|\{i \in H : p^*_i = F(p^*)\}| < |\{i \in H : p^*_i = q^*_0\}| = |\{i \in H : p^*_i = F'(p^*)\}|.$$

Then $F \notin M(\mathcal{F}^G_{\min})$ and the contradiction is found. □

Proposition 15. *Let $(p^j)^R_{j=1} \in \mathfrak{S}$ and f defined as in (2). Then $f\left(\times^R_{j=1} S^G_M(p^j)\right) = M(\mathcal{F}^G_{\min})$. In particular, $|M(\mathcal{F}^G_{\min})| = \prod^R_{j=1} |S^G_M(p^j)|$.*

Proof. In order to prove that $f\left(\times^R_{j=1} S^G_M(p^j)\right) \subseteq M(\mathcal{F}^G_{\min})$, let us fix $(q_j)^R_{j=1} \in \times^R_{j=1} S^G_M(p^j)$, define $F = f\left((q_j)^R_{j=1}\right)$ and prove that $F \in M(\mathcal{F}^G_{\min})$. By (3) we have $F \in \mathcal{F}^G_{\min}$. Given now $F' \in \mathcal{F}^G_{\min}$ and $p \in \mathcal{P}$, we get the proof showing that $|\{i \in H : p_i = F(p)\}| \geq |\{i \in H : p_i = F'(p)\}|$. Observe that $F(p), F'(p) \in S^G_2(p)$ and that there are $j \in \{1, \dots, R\}$ and $(\varphi, \psi, \rho) \in G$ such that $p = p^{j\,(\varphi,\psi,\rho)}$. Moreover, by Proposition 14, $F(p^j) = q_j \in S^G_M(p)$ while $F'(p^j) = q_j \in S^G_2(p)$ and thus $|\{i \in H : p^j_i = F(p^j)\}| \geq |\{i \in H : p^j_i = F'(p^j)\}|$. Note also that

$$|\{i \in H : p_i = F(p)\}| = |\{i \in H : p_i^{j\,(\varphi,\psi,\rho)} = F(p^{j\,(\varphi,\psi,\rho)})\}|$$

$$= |\{i \in H : \psi p^j_{\varphi^{-1}(i)}\rho = \psi F(p^j)\rho\}| = |\{i \in H : p^j_{\varphi^{-1}(i)} = F(p^j)\}| = |\{i \in H : p^j_i = F(p^j)\}|,$$

and, analogously

$$|\{i \in H : p_i = F'(p)\}| = |\{i \in H : p^j_i = F'(p^j)\}|.$$

Then, we finally get $|\{i \in H : p_i = F(p)\}| \geq |\{i \in H : p_i = F'(p)\}|$.

In order to prove that $M(\mathcal{F}^G_{\min}) \subseteq f\left(\times^R_{j=1} S^G_M(p^j)\right)$, observe that if $F \in M(\mathcal{F}^G_{\min})$, then we have that $F = \Psi\left[(p^j)^R_{j=1}, (F(p^j))^R_{j=1}\right]$ where, by Proposition 14, $(F(p^j))^R_{j=1} \in \times^R_{j=1} S^G_M(p^j)$. □

Proposition 16. *Let $F \in P(\mathcal{F}^G_{\min})$. Then, for every $p \in \mathcal{P}$, $F(p) \in S^G_P(p)$.*

Proof. Since $F \in \mathcal{F}^G_{\min}$, we know that, for every $p \in \mathcal{P}$, $F(p) \in S^G_2(p)$. Assume by contradiction there exists $p^* \in \mathcal{P}$ such that $F(p^*) \notin S^G_P(p^*)$. Then there exists $q^*_0 \in S^G_2(p^*)$ such that $A(p^*, q^*_0) > A(p^*, F(p^*))$. Consider now any $(p^j)^R_{j=1} \in \mathfrak{S}$ such that $p^1 = p^*$, and define $(q_j)^R_{j=1} \in \times^R_{j=1} S^G_2(p^j)$ as $q_1 = q^*_0$ and, for every $j \in \{2, \dots, R\}$, $q_j = F(p^j)$. Since the rule $F' = \Psi[(p^j)^R_{j=1}, (q_j)^R_{j=1}] \in \mathcal{F}^G_{\min}$ is such that $A(p^*, F'(p^*)) = A(p^*, q^*_0) > A(p^*, F(p^*))$, we get that $F \notin P(\mathcal{F}^G_{\min})$ and the contradiction is found. □

Proposition 17. *Let $(p^j)^R_{j=1} \in \mathfrak{S}$ and f defined as in (2). Then $f\left(\times^R_{j=1} S^G_P(p^j)\right) = P(\mathcal{F}^G_{\min})$. In particular, $|P(\mathcal{F}^G_{\min})| = \prod^R_{j=1} |S^G_P(p^j)|$.*

Proof. In order to prove that $f\left(\times^R_{j=1} S^G_P(p^j)\right) \subseteq P(\mathcal{F}^G_{\min})$, let us fix $(q_j)^R_{j=1} \in \times^R_{j=1} S^G_P(p^j)$, define $F = f\left((q_j)^R_{j=1}\right)$ and prove that $F \in P(\mathcal{F}^G_{\min})$. By (3) we have $F \in \mathcal{F}^G_{\min}$. Given now $F' \in \mathcal{F}^G_{\min}$ and $p \in \mathcal{P}$, we get the proof showing that $A(p, F'(p)) \not> A(p, F(p))$. Let $j \in \{1, \dots, R\}$ and $(\varphi, \psi, \rho) \in G$ such that $p = p^{j\,(\varphi,\psi,\rho)}$.Note that, $F(p^j) = q_j \in S^G_P(p)$ and $F'(p^j) \in S^G_2(p)$. Then, $A(p^j, F'(p^j)) \not> A(p^j, F(p^j))$. Using now Proposition 3, we have that for every x, y with $x \neq y$,

$$A(p, F(p), x, y) = A(p^{j\,(\varphi,\psi,\rho)}, \psi F(p^j)\rho, x, y) = A(p^j, F(p^j), \psi^{-1}(x), \psi^{-1}(y))$$

and

$$A(p, F'(p), x, y) = A(p^{j\,(\varphi,\psi,\rho)}, \psi F'(p^j)\rho, x, y) = A(p^j, F'(p^j), \psi^{-1}(x), \psi^{-1}(y)).$$

Recalling that ψ is a bijection, that implies $A(p, F'(p)) \not> A(p, F(p))$ and the proof is complete.

In order to prove that $P(\mathcal{F}^G_{\min}) \subseteq f\left(\times^R_{j=1} S^G_P(p^j)\right)$, observe that if $F \in P(\mathcal{F}^G_{\min})$, then we have that $F = \Psi\left[(p^j)^R_{j=1}, (F(p^j))^R_{j=1}\right]$ where, by Proposition 16, $(F(p^j))^R_{j=1} \in \times^R_{j=1} S^G_P(p^j)$. □

Proposition 18. *Let $F \in M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})$. Then, for every $p \in \mathcal{P}$, $F(p) \in S^G_M(p) \cap S^G_P(p)$.*

Proof. It immediately follows from Propositions 14 and 16. $\square$

Proposition 19. *Let $(p^j)_{j=1}^R \in \mathfrak{S}$ and f defined as in (2). Then $f\left(\times_{j=1}^R S^G_M(p^j) \cap S^G_P(p^j)\right) = M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})$. In particular, $|M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})| = \prod_{j=1}^R |S^G_M(p^j) \cap S^G_P(p^j)|$.*

Proof. In order to prove that $f\left(\times_{j=1}^R S^G_M(p^j) \cap S^G_P(p^j)\right) \subseteq M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})$, simply note that $f\left(\times_{j=1}^R S^G_M(p^j) \cap S^G_P(p^j)\right) \subseteq f\left(\times_{j=1}^R S^G_M(p^j)\right) \cap f\left(\times_{j=1}^R S^G_P(p^j)\right)$ and apply Propositions 15 and 17. In order to prove the opposite inclusion, observe that if $F \in M(\mathcal{F}^G_{\min}) \cap P(\mathcal{F}^G_{\min})$, then we have that $F = \Psi\left[(p^j)_{j=1}^R, (F(p^j))_{j=1}^R\right]$ where, by Proposition 18, $(F(p^j))_{j=1}^R \in \times_{j=1}^R S^G_M(p^j) \cap S^G_P(p^j)$. $\square$

Proposition 20. *Let $F \in P(M(\mathcal{F}^G_{\min}))$. Then, for every $p \in \mathcal{P}$, $F(p) \in S^G_{MP}(p)$.*

Proof. Follow the same argument as the proof of Proposition 16. $\square$

Proposition 21. *Let $(p^j)_{j=1}^R \in \mathfrak{S}$ and f defined in (2). Then $f\left(\times_{j=1}^R S^G_{MP}(p^j)\right) = P(M(\mathcal{F}^G_{\min}))$. In particular, $|P(M(\mathcal{F}^G_{\min}))| = \prod_{j=1}^R |S^G_{MP}(p^j)|$.*

Proof. Follow the same argument as the proof in Proposition 17. $\square$

Proposition 22. *Let $F \in M(P(\mathcal{F}^G_{\min}))$. Then, for every $p \in \mathcal{P}$, $F(p) \in S^G_{PM}(p)$.*

Proof. Follow the same argument as the proof of Proposition 14. $\square$

Proposition 23. *Let $(p^j)_{j=1}^R \in \mathfrak{S}$ and f defined as in (2). Then $f\left(\times_{j=1}^R S^G_{PM}(p^j)\right) = M(P(\mathcal{F}^G_{\min}))$. In particular, $|M(P(\mathcal{F}^G_{\min}))| = \prod_{j=1}^R |S^G_{PM}(p^j)|$.*

Proof. Follow the same argument as the proof of Proposition 15. $\square$

Proof of Theorem 1. Statement 1. Apply Propositions 12.1 and 15.

Statement 2. By Proposition 12.2, we know there exists $p \in \mathcal{P}$ such that $|S^G_M(p)| \geq 2$. Considering any $(p^j)_{j=1}^R \in \mathfrak{S}$ such that $p^1 = p$ and applying Proposition 15, we obtain the desired result.

Statement 3. Follow the same argument used to prove Statement 2 applying Propositions 12.3 and 17.

Statement 4. Apply Propositions 12.4, 12.6 and 19.

Statement 5. Follow the same argument used to prove Statement 2 applying Propositions 12.7 and 19.

Statement 6. Apply Propositions 12.4 and 21.

Statement 7. Apply Propositions 12.5 and 23.

Statement 8. Apply Propositions 12.6, 19, 21 and 23.

Statement 9. Follow the same argument used to prove Statement 2 applying Propositions 12.7, 21 and 23. $\square$

Proof of Theorem 2. Statement 1. By Proposition 13.1, we know there exists $p \in \mathcal{P}$ such that $|S^G_M(p) \cap S^G_P(p)| = 0$. Considering any $(p^j)_{j=1}^R \in \mathfrak{S}$ such that $p^1 = p$ and applying Propositions 19, we obtain the desired result.

Statement 2. Follow the same argument used to prove Statement 1 applying Propositions 9, 13.2 and 21.

Statement 3. Follow the same argument used to prove Statement 1 applying Propositions 10, 13.2 and 23. $\square$

6 Proof of Proposition 12

6.1 Preliminary notation and remarks.

We assume $n = 3$ and $\gcd(h, n!) = 1$. For every $p \in \mathcal{P}$, define

$$\begin{aligned} v_1(p) &= \left|\left\{i \in H : p_i = [1,2,3]^T\right\}\right|, \quad v_2(p) = \left|\left\{i \in H : p_i = [1,3,2]^T\right\}\right|, \\ v_3(p) &= \left|\left\{i \in H : p_i = [2,1,3]^T\right\}\right|, \quad v_4(p) = \left|\left\{i \in H : p_i = [2,3,1]^T\right\}\right|, \\ v_5(p) &= \left|\left\{i \in H : p_i = [3,1,2]^T\right\}\right|, \quad v_6(p) = \left|\left\{i \in H : p_i = [3,2,1]^T\right\}\right|, \end{aligned} \tag{4}$$

and note that $(v_1(p), \dots, v_6(p)) \in \mathbb{N}_0^6$ and that $\sum_{j=1}^{6} v_j(p) = h$. Define also, for every $x, y \in \{1,2,3\}$ with $x \neq y$,

$$s_{x,y}(p) = |\{i \in H : (x,y) \in p_i\}|, \tag{5}$$

and note that $s_{x,y}(p) = h - s_{y,x}(p)$ and

$$s_{1,2}(p) = v_1(p) + v_2(p) + v_5(p), \quad s_{2,3}(p) = v_1(p) + v_3(p) + v_4(p), \quad s_{3,1}(p) = v_4(p) + v_5(p) + v_6(p),$$

$$s_{2,1}(p) = v_3(p) + v_4(p) + v_6(p), \quad s_{3,2}(p) = v_2(p) + v_5(p) + v_6(p), \quad s_{1,3}(p) = v_1(p) + v_2(p) + v_3(p).$$

We also know that[8] if $(\varphi, \psi, \rho_0) \in \mathrm{Stab}_G(p)$ and $(\varphi', \psi', \rho_0) \in \mathrm{Stab}_G(p)$, then $\psi = \psi'$ and ψ is a conjugate of $\rho_0 = (13)$ depending on p, that is, $\psi \in \{(12), (13), (23)\}$. Using the following computations

$$(12)\begin{bmatrix}1\\2\\3\end{bmatrix}\rho_0 = \begin{bmatrix}3\\1\\2\end{bmatrix}, \quad (12)\begin{bmatrix}1\\3\\2\end{bmatrix}\rho_0 = \begin{bmatrix}1\\3\\2\end{bmatrix}, \quad (12)\begin{bmatrix}2\\1\\3\end{bmatrix}\rho_0 = \begin{bmatrix}3\\2\\1\end{bmatrix}, \quad (12)\begin{bmatrix}2\\3\\1\end{bmatrix}\rho_0 = \begin{bmatrix}2\\3\\1\end{bmatrix}, \quad (12)\begin{bmatrix}3\\1\\2\end{bmatrix}\rho_0 = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad (12)\begin{bmatrix}3\\2\\1\end{bmatrix}\rho_0 = \begin{bmatrix}2\\1\\3\end{bmatrix},$$

$$(13)\begin{bmatrix}1\\2\\3\end{bmatrix}\rho_0 = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad (13)\begin{bmatrix}1\\3\\2\end{bmatrix}\rho_0 = \begin{bmatrix}2\\1\\3\end{bmatrix}, \quad (13)\begin{bmatrix}2\\1\\3\end{bmatrix}\rho_0 = \begin{bmatrix}1\\3\\2\end{bmatrix}, \quad (13)\begin{bmatrix}2\\3\\1\end{bmatrix}\rho_0 = \begin{bmatrix}3\\1\\2\end{bmatrix}, \quad (13)\begin{bmatrix}3\\1\\2\end{bmatrix}\rho_0 = \begin{bmatrix}2\\3\\1\end{bmatrix}, \quad (13)\begin{bmatrix}3\\2\\1\end{bmatrix}\rho_0 = \begin{bmatrix}3\\2\\1\end{bmatrix},$$

$$(23)\begin{bmatrix}1\\2\\3\end{bmatrix}\rho_0 = \begin{bmatrix}2\\3\\1\end{bmatrix}, \quad (23)\begin{bmatrix}1\\3\\2\end{bmatrix}\rho_0 = \begin{bmatrix}3\\2\\1\end{bmatrix}, \quad (23)\begin{bmatrix}2\\1\\3\end{bmatrix}\rho_0 = \begin{bmatrix}2\\1\\3\end{bmatrix}, \quad (23)\begin{bmatrix}2\\3\\1\end{bmatrix}\rho_0 = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad (23)\begin{bmatrix}3\\1\\2\end{bmatrix}\rho_0 = \begin{bmatrix}3\\1\\2\end{bmatrix}, \quad (23)\begin{bmatrix}3\\2\\1\end{bmatrix}\rho_0 = \begin{bmatrix}1\\3\\2\end{bmatrix},$$

it is simple to prove the following statements:

- there exists $\varphi \in S_h$ such that $(\varphi, (12), \rho_0) \in \mathrm{Stab}_G(p)$ if and only if $v_1(p) = v_5(p)$ and $v_3(p) = v_6(p)$. In that case $S_1^G(p) = \left\{[1,3,2]^T, [2,3,1]^T\right\}$.
- there exists $\varphi \in S_h$ such that $(\varphi, (13), \rho_0) \in \mathrm{Stab}_G(p)$ if and only if $v_2(p) = v_3(p)$ and $v_4(p) = v_5(p)$. In that case $S_1^G(p) = \left\{[1,2,3]^T, [3,2,1]^T\right\}$.
- there exists $\varphi \in S_h$ such that $(\varphi, (23), \rho_0) \in \mathrm{Stab}_G(p)$ if and only if $v_1(p) = v_4(p)$ and $v_2(p) = v_6(p)$. In that case $S_1^G(p) = \left\{[2,1,3]^T, [3,1,2]^T\right\}$.

As a consequence,

$$\begin{aligned} &(v_1(p) \neq v_5(p) \text{ or } v_3(p) \neq v_6(p)) \text{ and} \\ &(v_2(p) \neq v_3(p) \text{ or } v_4(p) \neq v_5(p)) \text{ and} \\ &(v_1(p) \neq v_4(p) \text{ or } v_2(p) \neq v_6(p)), \end{aligned} \tag{6}$$

is a necessary a sufficient condition to have $S_1^G(p) = \mathcal{L}(N)$.

Fixed $p \in \mathcal{P}$ and $j \in \{2, M, P, MP, PM\}$, let us consider now the problem to compute $S_j^G(p)$. First of all, let us observe that, by Proposition 4, we can assume without loss of generality that

$$v_1(p) \geq v_j(p) \ \text{ for all } \ j \in \{1, \dots, 6\}. \tag{7}$$

Define

$$\nu_h = \frac{h+1}{2},$$

[8] See Lemma 15 and Theorem 6 in Bubboloni and Gori (2014).

and note that $\nu(p) \in \{\nu_h, \ldots, h\}$. Moreover, for every $x, y \in N$ with $x \neq y$, either $s_{x,y}(p) \geq \nu_h$ or $s_{y,x}(p) \geq \nu_h$. As a consequence, $C_{\nu_h}(p) \neq \varnothing$ implies $|C_{\nu_h}(p)| = 1$ and $\nu(p) = \nu_h$.

Consider now $p \in \mathcal{P}$ and $\nu \in \{\nu_h, \ldots, h\}$ and assume that p satisfies (7). Then p cannot be solution to the system

$$\begin{cases} s_{1,3}(p) \geq \nu \\ s_{3,2}(p) \geq \nu \\ s_{2,1}(p) \geq \nu \end{cases} \tag{8}$$

Indeed, (7) and the second and third equation in (8) imply that

$$h = (v_2(p) + v_5(p) + v_6(p)) + (v_3(p) + v_4(p) + v_1(p)) \geq$$

$$(v_2(p) + v_5(p) + v_6(p)) + (v_3(p) + v_4(p) + v_6(p)) = s_{3,2}(p) + s_{2,1}(p) \geq 2\nu > h,$$

that is, a contradiction. As a consequence, if p satisfies (7), then we have that $C_\nu(p) = \varnothing$ is equivalent to require that p solves the system

$$\begin{cases} s_{1,2}(p) \geq \nu \\ s_{2,3}(p) \geq \nu \\ s_{3,1}(p) \geq \nu \end{cases}$$

6.2 Case-by-case analysis

Let us fix $p \in \mathcal{P}$ satisfying (7). Our purpose is the computation of the sets $S_2^G(p)$, $S_M^G(p)$, $S_P^G(p)$, $S_M^G(p) \cap S_P^G(p)$, $S_{MP}^G(p)$ and $S_{PM}^G(p)$. We write v_j instead of $v_j(p)$ and $s_{x,y}$ instead of $s_{x,y}(p)$.

Case 1. Assume that $v_1 = v_5$ and $v_3 = v_6$. Observe that $s_{2,3} = s_{3,1}$. Note also that it has to be $v_2 \neq v_4$ else

$$h = \sum_{j=1}^{6} v_j = 2v_1 + 2v_2 + 2v_3,$$

that implies the contradiction $2 \mid h$. Note also that $s_{2,3} = s_{1,2}$ implies $v_4 > v_2$. Indeed, $s_{2,3} = s_{1,2}$ is the same that $v_1 + v_3 + v_4 = 2v_1 + v_2$, that is, $v_3 + v_4 = v_1 + v_2$. As $v_1 \geq v_3$ and $v_2 \neq v_4$ it has to be $v_4 > v_2$. There are several cases to discuss.

Case 1.1. If $s_{2,3} < h/2$, then it has to be $s_{1,2} > h/2$ as p does not solve (8). Then $C_{\nu_h}(p) = \{[1,3,2]^T\}$. That implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[1,3,2]^T\}$.

Case 1.2. If $s_{2,3} > h/2$ and $s_{1,2} < h/2$, then $C_{\nu_h}(p) = \{[2,3,1]^T\}$. That implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[2,3,1]^T\}$.

Case 1.3. If $s_{2,3} > s_{1,2} > h/2$, then $C_{s_{1,2}}(p) = \varnothing$ and $C_{s_{1,2}+1}(p) = \{[2,3,1]^T\}$. That implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[2,3,1]^T\}$.

Case 1.4. If $s_{2,3} = s_{1,2} > h/2$, then $C_{s_{1,2}}(p) = \varnothing$ and $C_{s_{1,2}+1}(p) = \mathcal{L}(N)$. Then $S_2^G(p) = \{[1,3,2]^T, [2,3,1]^T\}$ and $S_P^G(p) = \{[1,3,2]^T, [2,3,1]^T\}$. Since the equality $s_{2,3} = s_{1,2}$ implies $v_4 > v_2$, we get $S_M^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[2,3,1]\}$.

Case 1.5. If $s_{1,2} > s_{2,3} > h/2$, then $C_{s_{2,3}}(p) = \varnothing$ and $C_{s_{2,3}+1}(p) = \{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$. Then $S_2^G(p) = \{[1,3,2]^T\}$ and that implies $S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[1,3,2]^T\}$.

Case 2. Assume that $v_2 = v_3$ and $v_4 = v_5$. Observe that $s_{1,2} = s_{2,3}$. Note also that it has to be $v_1 \neq v_6$ else

$$h = \sum_{j=1}^{6} v_j = 2v_1 + 2v_2 + 2v_3,$$

that implies the contradiction $2 \mid h$. As a consequence $v_1 > v_6$. That implies that $s_{1,2} + s_{2,3} > h$ and then $s_{1,2} > h/2$. There are several cases to discuss.

Case 2.1. If $s_{3,1} < h/2$, then $C_{\nu_h}(p) = \{[1,2,3]^T\}$. That implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[1,3,2]^T\}$.

Case 2.2. If $s_{1,2} > s_{3,1} > h/2$, then $C_{s_{3,1}}(p) = \varnothing$ and $C_{s_{3,1}+1}(p) = \{[1,2,3]^T\}$. That implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[1,2,3]^T\}$.

Case 2.3. If $s_{1,2} = s_{3,1} > h/2$, then $C_{s_{3,1}}(p) = \varnothing$ and $C_{s_{3,1}+1}(p) = \mathcal{L}(N)$. Then $S_2^G(p) = \{[1,2,3]^T, [3,2,1]^T\}$ and $S_P^G(p) = \{[1,2,3]^T, [3,2,1]^T\}$. Since $v_1 > v_6$, we get $S_M^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[1,2,3]\}$.

Case 2.4. If $s_{3,1} > s_{1,2} > h/2$, then $C_{s_{1,2}}(p) = \varnothing$ and $C_{s_{1,2}+1}(p) = \{[2,3,1]^T, [3,1,2]^T, [3,2,1]^T\}$. Then $S_2^G(p) = \{[3,2,1]^T\}$ and that implies $S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[3,2,1]^T\}$.

Case 3. Assume that $v_1 = v_4$ and $v_2 = v_6$. Observe that $s_{1,2} = s_{3,1}$. Note also that it has to be $v_3 \neq v_5$ else

$$h = \sum_{j=1}^{6} v_j = 2v_1 + 2v_2 + 2v_3,$$

that implies the contradiction $2 \mid h$. Note also that $s_{2,3} = s_{1,2}$ implies $v_5 > v_3$. Indeed $s_{2,3} = s_{1,2}$ is the same that $2v_1 + v_3 = v_1 + v_2 + v_5$, that is, $v_1 + v_3 = v_2 + v_5$. As $v_1 \geq v_2$ and $v_3 \neq v_5$ it has to be $v_5 > v_3$. There are several cases to discuss.

Case 3.1. If $s_{1,2} < h/2$, then it has to be $s_{2,3} > h/2$ as p does not solve (8). Then $C_{\nu_h}(p) = \{[2,1,3]^T\}$. That implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[2,1,3]^T\}$.

Case 3.2. If $s_{1,2} > h/2$ and $s_{2,3} < h/2$, then $C_{\nu_h}(p) = \{[3,1,2]^T\}$. That implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[3,1,2]^T\}$.

Case 3.3. If $s_{1,2} > s_{2,3} > h/2$, then $C_{s_{2,3}}(p) = \varnothing$ and $C_{s_{2,3}+1}(p) = \{[3,1,2]^T\}$. That implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[3,1,2]^T\}$.

Case 3.4. If $s_{1,2} = s_{2,3} > h/2$, then $C_{s_{2,3}}(p) = \varnothing$ and $C_{s_{2,3}+1}(p) = \mathcal{L}(N)$. Then $S_2^G(p) = \{[2,1,3]^T, [3,1,2]^T\}$ and $S_P^G(p) = \{[2,1,3]^T, [3,1,2]^T\}$. Since the equality $s_{1,2} = s_{2,3}$ implies $v_5 > v_3$, we get $S_M^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[3,1,2]\}$.

Case 3.5. If $s_{2,3} > s_{1,2} > h/2$, then $C_{s_{1,2}}(p) = \varnothing$ and $C_{s_{1,2}+1}(p) = \{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$. Then $S_2^G(p) = \{[2,1,3]^T\}$ and that implies $S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[2,1,3]^T\}$.

Case 4. Assume (6) so that $S_1^G(p) = \mathcal{L}(N)$. If $C_{\nu_h}(p) \neq \varnothing$, then $C_{\nu_h}(p) = S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p)$ and those sets are all singletons. If instead $C_{\nu_h}(p) = \varnothing$, since $C_h(p) \neq \varnothing$, then there exists[9] $\nu^* \in \{\nu_h, \dots, h-1\}$ such that $C_{\nu^*}(p) = \varnothing$ and $C_{\nu^*+1}(p) \neq \varnothing$. Thus, p solves the system

$$\begin{cases} s_{1,2} \geq \nu^* \\ s_{2,3} \geq \nu^* \\ s_{3,1} \geq \nu^* \end{cases}$$

where at least one of the inequalities is indeed an equality. Then, we need to refine the discussion introducing further cases.

Case 4.1. Assume that p solves

$$\begin{cases} s_{1,2} \geq \nu^* + 1 \\ s_{2,3} \geq \nu^* + 1 \\ s_{3,1} = \nu^* \end{cases}$$

Then $C_{\nu^*+1}(p) = \{[1,2,3]^T\}$ and that implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[1,2,3]^T\}$.

[9]Note that $\nu_h < h-1$ if and only if $h > 3$. Since $\gcd(h,6) = 1$, that condition is satisfied.

Case 4.2. Assume that p solves

$$\begin{cases} s_{1,2} \geq \nu^* + 1 \\ s_{2,3} = \nu^* \\ s_{3,1} \geq \nu^* + 1 \end{cases}$$

Then $C_{\nu^*+1}(p) = \{[3,1,2]^T\}$ and that implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[3,1,2]^T\}$.

Case 4.3. Assume that p solves

$$\begin{cases} s_{1,2} = \nu^* \\ s_{2,3} \geq \nu^* + 1 \\ s_{3,1} \geq \nu^* + 1 \end{cases}$$

Then $C_{\nu^*+1}(p) = \{[2,3,1]^T\}$ and that implies $S_2^G(p) = S_M^G(p) = S_P^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[2,3,1]^T\}$.

Case 4.4. Assume that p solves

$$\begin{cases} s_{1,2} \geq \nu^* + 1 \\ s_{2,3} = \nu^* \\ s_{3,1} = \nu^* \end{cases}$$

Then $C_{\nu^*+1}(p) = S_2^G(p) = \{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$ and $S_P^G(p) = \{[1,2,3]^T, [3,1,2]^T\}$. Let us compare now v_1, v_2 and v_5. Surely we have that $v_1 \geq v_2, v_5$. Note that it cannot be $v_1 = v_5$ because from the equality $s_{2,3} = s_{3,1}$ we get $v_3 = v_6$ and (6) is violated. Assume now by contradiction that $v_1 = v_2$. From

$$\sum_{j=1}^{6} v_j = h, \quad (v_1 + v_3 + v_4) + (v_4 + v_5 + v_6) = 2\nu^*,$$

we get $v_4 - v_2 = 2\nu^* - h$. Since $v_1 = v_2$ and $2\nu^* - h \geq 1$, we have $v_4 \geq v_1 + 1$ and the contradiction is found. As a consequence, $v_1 > v_2, v_5$ and then $S_M^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[1,2,3]^T\}$.

Case 4.5. Assume that p solves

$$\begin{cases} s_{1,2} = \nu^* \\ s_{2,3} \geq \nu^* + 1 \\ s_{3,1} = \nu^* \end{cases}$$

Then $C_{\nu^*+1}(p) = S_2^G(p) = \{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$ and $S_P^G(p) = \{[1,2,3]^T, [2,3,1]^T\}$. Let us compare now v_1, v_3 and v_4. Surely we have that $v_1 \geq v_3, v_4$. Note that it cannot be $v_1 = v_4$ because from the equality $s_{1,2} = s_{3,1}$ we get $v_2 = v_6$ and (6) is violated. Assume now by contradiction that $v_1 = v_3$. From

$$\sum_{j=1}^{6} v_j = h, \quad (v_1 + v_2 + v_5) + (v_4 + v_5 + v_6) = 2\nu^*,$$

we get $v_5 - v_3 = 2\nu^* - h$. Since $v_1 = v_3$ and $2\nu^* - h \geq 1$, we have $v_5 \geq v_1 + 1$ and the contradiction is found. As a consequence, $v_1 > v_3, v_4$ and then $S_M^G(p) = S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p) = \{[1,2,3]^T\}$,

Case 4.6. Assume that p solves

$$\begin{cases} s_{1,2} = \nu^* \\ s_{2,3} = \nu^* \\ s_{3,1} \geq \nu^* + 1 \end{cases} \tag{9}$$

Then $C_{\nu^*+1}(p) = S_2^G(p) = \{[2,3,1]^T, [3,1,2]^T, [3,2,1]^T\}$ and a simple computation shows that $S_P^G(p) = \{[2,3,1]^T, [3,1,2]^T\}$. Let us compare now v_4, v_5 and v_6. Note also that it cannot be $v_4 = v_5$ because from the equality $s_{1,2} = s_{2,3}$ we get $v_2 = v_3$ and (6) is violated. Thus, there are six further sub-cases to analyze.

Case 4.6.1. if $v_4 > v_5, v_6$ then $S^G_M(p) = S^G_M(p) \cap S^G_P(p) = S^G_{MP}(p) = S^G_{PM}(p) = \{[2,3,1]^T\}$,

Case 4.6.2. if $v_5 > v_4, v_6$ then $S^G_M(p) = S^G_M(p) \cap S^G_P(p) = S^G_{MP}(p) = S^G_{PM}(p) = \{[3,1,2]^T\}$,

Case 4.6.3. if $v_6 > v_4 > v_5$ then $S^G_M(p) = S^G_{MP}(p) = \{[3,2,1]^T\}$, $S^G_M(p) \cap S^G_P(p) = \varnothing$, and $S^G_{PM}(p) = \{[2,3,1]^T\}$.

Case 4.6.4. if $v_6 > v_5 > v_4$ then $S^G_M(p) = S^G_{MP}(p) = \{[3,2,1]^T\}$, $S^G_M(p) \cap S^G_P(p) = \varnothing$, and $S^G_{PM}(p) = \{[3,1,2]^T\}$.

Case 4.6.5. if $v_4 = v_6 > v_5$, then $S^G_M(p) = \{[2,3,1]^T, [3,2,1]^T\}$. That implies that $S^G_M(p) \cap S^G_P(p) = \{[2,3,1]^T\}$, and $S^G_{MP}(p) = S^G_{PM}(p) = \{[2,3,1]^T\}$.

Case 4.6.6. if $v_5 = v_6 > v_4$, then $S^G_M(p) = \{[3,1,2]^T, [3,2,1]^T\}$. That implies that $S^G_M(p) \cap S^G_P(p) = \{[3,1,2]^T\}$, and $S^G_{MP}(p) = S^G_{PM}(p) = \{[3,1,2]^T\}$.

Case 4.7 Assume that p solves

$$\begin{cases} s_{1,2} = \nu^* \\ s_{2,3} = \nu^* \\ s_{3,1} = \nu^* \end{cases}$$

Then $C_{\nu^*+1}(p) = S^G_2(p) = \mathcal{L}(N)$ and $S^G_P(p) = \{[1,2,3]^T, [2,3,1]^T, [3,1,2]^T\}$. Assume by contradiction there is $j^* \in \{2,\ldots,6\}$ such that $v_1 = v_{j^*}$. From the system above we deduce that

$$v_1 - v_6 = v_4 - v_2 = v_5 - v_3.$$

If $j^* \in \{2,3,6\}$, then we have $v_1 = v_6$, $v_4 = v_2$ and $v_5 = v_3$ and that implies the contradiction $2 \mid h$. If $j^* \in \{4,5\}$, then (6) is violated and the contradiction is found. Then $v_1 > v_j$ for all $j \in \{2,\ldots,6\}$, and that implies $S^G_M(p) = S^G_M(p) \cap S^G_P(p) = S^G_{MP}(p) = S^G_{PM}(p) = \{[1,2,3]^T\}$.

6.3 Last part of the proof

Statements 1 and 2. By Section 6.2 and using Proposition 4, we deduce that there exists $p \in \mathcal{P}$ such that $|S^G_M(p)| \geq 2$ if and only if, according to the Cases 4.6.5 and 4.6.6, there exist $(v_1,\ldots,v_6) \in \mathbb{N}_0^6$ and $\nu^* \in \{\nu_h,\ldots,h-1\}$ such that

a) $v_1 \geq v_j$ for all $j \in \{1,\ldots,6\}$,

b) $\sum_{j=1}^6 v_j = h$,

c) (6) and (9) hold true,

d) $v_4 = v_6 > v_5$ or $v_5 = v_6 > v_4$.

Since $h \in A \cup B$, where

$$A = \{5,7,11\}, \quad B = \{13 + 6k + 4r : k \in \mathbb{N}_0, r \in \{0,1\}\},$$

we are left with showing that there exist $(v_1,\ldots,v_6)$ and ν^* satisfying a), b), c) and d) if and only if $h \in B$. If $h = 13 + 6k + 4r$ for some $k \in \mathbb{N}_0$ and $r \in \{0,1\}$, then $\nu_h = 7 + 3k + 2r$ and a simple check shows that

$$(v_1,v_2,v_3,v_4,v_5,v_6) = (4+k+r, k, 1+k, 2+k+r, 3+k+r, 3+k+r), \quad \nu^* = \nu_h$$

satisfy a), b), c) and d) with $v_5 = v_6$.

Consider then $h \in \{5,7,11\}$ and assume by contradiction there exist $(v_1,\ldots,v_6)$ and ν^* satisfying a), b), c) and $v_4 = v_6 > v_5$. A similar argument works also if $v_5 = v_6 > v_4$. From b) and (9) we get $v_1 - v_6 = 2\nu^* - h \geq 1$. Then there exists, $t \in \mathbb{N}_0$ such that $v_6 = t$, $v_4 = t$ and $v_1 = t + (2\nu^* - h)$. Moreover, since $v_5 < v_6$, we also have that $v_5 \leq t - 1$. We have then

$$3t - 1 \geq v_4 + v_5 + v_6 = s_{3,1} \geq \nu^* + 1 \geq \frac{h+1}{2} + 1,$$

that is, $t \geq \frac{h+5}{6}$, and also

$$h \geq v_1 + v_4 + v_6 = 3t + 2\nu^* - h \geq 3t + 1,$$

that is, $t \leq \frac{h-1}{3}$. Let us discuss now the three possible values of h.

- If $h = 5$, then $\frac{h+5}{6} > \frac{h-1}{3}$ and the contradiction follows.
- If $h = 7$, then $\nu^* \in \{4, 5, 6\}$, $t = 2$ and $v_1 + v_4 + v_6 \geq 7$. As a consequence, $v_5 = 0$ and $s_{3,1} = v_4 + v_5 + v_6 = 4 \notin \{5, 6, 7\}$ so that the contradiction follows.
- If $h = 11$, then $\nu^* \in \{6, 7, 8, 9, 10\}$ and $t = 3$. If $\nu^* = 6$, then $v_4 = v_6 = 3$ and $v_1 = 4$. Since $s_{3,1} = v_4 + v_5 + v_6 \geq \nu^* + 1 = 7$, it has to be $v_2 = v_3 = 0$ and $v_5 = 1$. As a consequence, $s_{1,2} = v_1 + v_2 + v_5 = 5 < \nu^*$ and the contradiction is found. If $\nu^* \geq 7$, then $v_1 + v_4 + v_6 \geq 12$ and the contradiction follows.

Statement 3. By Section 6.2 and Proposition 4, we deduce that there exists $p \in \mathcal{P}$ such that $|S_P^G(p)| \geq 2$ if, according to Case 2.3, there exist $(v_1, \ldots, v_6) \in \mathbb{N}_0^6$ and $\nu^* \in \{\nu_h, \ldots, h-1\}$ such that

a) $v_1 \geq v_j$ for all $j \in \{1, \ldots, 6\}$,

b) $\sum_{j=1}^{6} v_j = h$,

c) $v_2 = v_3$, $v_4 = v_5$, and $v_1 + v_2 + v_5 = v_4 + v_5 + v_6 > h/2$,

It can be immediately checked that if $h = 5 + 6k + 2r$ for some $k \in \mathbb{N} \cup \{0\}$ and $r \in \{0, 1\}$, then we have that

$$(v_1, v_2, v_3, v_4, v_5, v_6) = (2 + k + r, k, k, 1 + k, 1 + k, 1 + k + r)$$

satisfy a), b), and c).

Statements 4 and 5. They follow from Section 6.2 and Proposition 4.

Statements 6 and 7. By Section 6.2 and Proposition 4, we deduce that $S_M^G(p) \cap S_P^G(p) = S_{MP}^G(p) = S_{PM}^G(p)$ for all $p \in \mathcal{P}$ if, according to Cases 4.6.3 and 4.6.4, there are no $(v_1, \ldots, v_6) \in \mathbb{N}_0^6$ and $\nu^* \in \{\nu_h, \ldots, h-1\}$ such that

a) $v_1 \geq v_j$ for all $j \in \{1, \ldots, 6\}$,

b) $\sum_{j=1}^{6} v_j = h$,

c) (6) and (9) hold true,

d) $v_6 > v_4, v_5$,

while there exists $p \in \mathcal{P}$ such that $S_M^G(p) \cap S_P^G(p) = \varnothing$ and $S_{MP}^G(p) \neq S_{PM}^G(p)$ otherwise.

We have that $h \in A \cup B$, where

$$A = \{5, 7, 11, 13\}, \quad B = \{17 + 6k + 2r : k \in \mathbb{N}_0, r \in \{0, 1\}\},$$

so that we are left with showing that there exist $(v_1, \ldots, v_6)$ and ν^* satisfying a), b), c) and d) if and only if $h \in B$. A simple computation show that if $h = 17 + 6k + 2r$ for some $k \in \mathbb{N}_0$ and $r \in \{0, 1\}$, then $\nu_h = 9 + 3k + r$ and we have that

$$(v_1, v_2, v_3, v_4, v_5, v_6) = (6 + k, k, 1 + k, 2 + k + r, 3 + k + r, 5 + k), \quad \nu^* = \nu_h \tag{10}$$

satisfy a), b), c) and d).

Consider then $h \in \{5, 7, 11, 13\}$. Assume by contradiction there exist $(v_1, \ldots, v_6)$ and ν^* satisfying a), b), c) and d). From b) and (9) we get

$$v_5 - v_3 \geq 2\nu^* - h + 1 \geq 2, \quad v_4 - v_2 \geq 2\nu^* - h + 1 \geq 2, \quad v_1 - v_6 = 2\nu^* - h \geq 1$$

as $\nu^* \geq \nu_h$. Moreover, from (6) and (9), we also have that both $v_2 \neq v_3$ and $v_4 \neq v_5$. Assuming then that $v_2 < v_3$, using a) and d) we obtain

$$v_2 \geq 0, \quad v_3 \geq 1, \quad v_4 \geq v_2 + 2 \geq 2, \quad v_5 \geq v_3 + 2 \geq 3, \quad v_6 \geq v_5 + 1 \geq 4, \quad v_1 \geq v_6 + 1 \geq 5$$

As a consequence, $h = \sum_{j=1}^{6} v_j \geq 15$ and the contradiction is found. If $v_3 < v_2$, a similar argument leads again to a contradiction.

7 Proof of Proposition 13

Statement 1. We have that $h \in \{5, 7, 11, 13\}$ or $h = 17 + 6k + 2r$ for some $k \in \mathbb{N}_0$ and $r \in \{0, 1\}$.

If $h = 5$, then $n = 4$. Define $p \in \mathcal{P}$ as

$$p = \begin{bmatrix} 1 & 1 & 3 & 4 & 4 \\ 2 & 2 & 4 & 2 & 3 \\ 3 & 3 & 2 & 1 & 2 \\ 4 & 4 & 1 & 3 & 1 \end{bmatrix},$$

A computation shows that $\mathrm{Stab}_G(p) \leq S_h \times \{id\} \times \{id\}$ so that $S_1^G(p) = \mathcal{L}(N)$. Moreover, $\nu(p) = 4$ and $S_2^G(p) = C_4(p) = \mathcal{L}(N)$. As a consequence, $S_M^G(p) = \{[1, 2, 3, 4]^T\}$. However, it is easily checked that

$$A(p, [2, 1, 3, 4]^T) > A(p, [1, 2, 3, 4]^T)$$

so that $[1, 2, 3, 4]^T \notin S_P^G(p)$ and $S_M^G(p) \cap S_P^G(p) = \varnothing$.

If $h = 7$, then $n \in \{4, 5, 6\}$. Define $p \in \mathcal{P}$ as

$$p = \begin{bmatrix} 1 & 1 & 2 & 1 & 4 & 3 & 4 \\ 2 & 2 & 4 & 3 & 1 & 2 & 3 \\ 3 & 3 & 1 & 2 & 3 & 4 & 1 \\ 4 & 4 & 3 & 4 & 2 & 1 & 2 \\ (5) & (5) & (5) & (5) & (5) & (5) & (5) \\ (6) & (6) & (6) & (6) & (6) & (6) & (6) \end{bmatrix},$$

where the last rows with entries in brackets have to be added according to n. A computation shows that $\mathrm{Stab}_G(p) \leq S_h \times \{id\} \times \{id\}$ so that $S_1^G(p) = \mathcal{L}(N)$. Moreover, $\nu(p) = 5$ and

$$S_2^G(p) = C_5(p) = \{[1, 3, 2, 4, (5), (6)]^T,\ [1, 2, 3, 4, (5), (6)]^T,\ [1, 2, 4, 3, (5), (6)]^T\},$$

$$S_M^G(p) = \{[1, 2, 3, 4, (5), (6)]^T\}, \quad \text{and} \quad S_P^G(p) = \{[1, 3, 2, 4, (5), (6)]^T\},$$

so that $S_M^G(p) \cap S_P^G(p) = \varnothing$.

If $h = 11$, then $n \in \{4, \dots, 10\}$. Define $p \in \mathcal{P}$ as

$$p = \begin{bmatrix} 1 & 1 & 2 & 1 & 4 & 3 & 4 & 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 3 & 1 & 2 & 3 & 2 & 1 & 4 & 3 \\ 3 & 3 & 1 & 2 & 3 & 4 & 1 & 3 & 4 & 1 & 2 \\ 4 & 4 & 3 & 4 & 2 & 1 & 2 & 4 & 3 & 2 & 1 \\ (5) & (5) & (5) & (5) & (5) & (5) & (5) & (5) & (5) & (5) & (5) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (10) & (10) & (10) & (10) & (10) & (10) & (10) & (10) & (10) & (10) & (10) \end{bmatrix},$$

and note that $\mathrm{Stab}_G(p) \leq S_h \times \{id\} \times \{id\}$ so that $S_1^G(p) = \mathcal{L}(N)$. Moreover, $\nu(p) = 7$ and

$$S_2^G(p) = C_7(p) = \{[1, 3, 2, 4, (5), \dots, (10)]^T,\ [1, 2, 3, 4, (5), \dots, (10)]^T,\ [1, 2, 4, 3, (5), \dots, (10)]^T\},$$

$$S_M^G(p) = \{[1, 2, 3, 4, (5), \dots, (10)]^T\}, \quad \text{and} \quad S_P^G(p) = \{[1, 3, 2, 4, (5), \dots, (10)]^T\},$$

so that $S_M^G(p) \cap S_P^G(p) = \varnothing$.

If $h = 13$, then $n \in \{4, \dots, 12\}$. Define $p \in \mathcal{P}$ as

$$p = \begin{bmatrix} 1 & 1 & 2 & 1 & 4 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 4 \\ 2 & 2 & 4 & 3 & 1 & 2 & 3 & 2 & 1 & 4 & 3 & 2 & 3 \\ 3 & 3 & 1 & 2 & 3 & 4 & 1 & 3 & 4 & 1 & 2 & 3 & 2 \\ 4 & 4 & 3 & 4 & 2 & 1 & 2 & 4 & 3 & 2 & 1 & 4 & 1 \\ (5) & (5) & (5) & (5) & (5) & (5) & (5) & (5) & (5) & (5) & (5) & (5) & (5) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (12) & (12) & (12) & (12) & (12) & (12) & (12) & (12) & (12) & (12) & (12) & (12) & (12) \end{bmatrix},$$

and note that $\mathrm{Stab}_G(p) \leq S_h \times \{id\} \times \{id\}$ so that $S_1^G(p) = \mathcal{L}(N)$. Moreover, $\nu(p) = 8$ and

$$S_2^G(p) = C_8(p) = \{[1,3,2,4,(5),\dots,(12)]^T,\ [1,2,3,4,(5),\dots,(12)]^T,\ [1,2,4,3,(5),\dots,(12)]^T\},$$

$$S_M^G(p) = \{[1,2,3,4,(5),\dots,(12)]^T\}, \quad \text{and} \quad S_P^G(p) = \{[1,3,2,4,(5),\dots,(12)]^T\},$$

so that $S_M^G(p) \cap S_P^G(p) = \varnothing$.

Finally assume that $h = 17+6k+2r$ for some $k \in \mathbb{N}_0$ and $r \in \{0,1\}$. Consider then any preference profile $p \in \mathcal{P}$ such that

$$\begin{aligned}
&|\{i \in H : p_i = [1,2,3,4,\dots,n]^T\}| = 6+k,\\
&|\{i \in H : p_i = [1,3,2,4,\dots,n]^T\}| = k,\\
&|\{i \in H : p_i = [2,1,3,4,\dots,n]^T\}| = 1+k,\\
&|\{i \in H : p_i = [2,3,1,4,\dots,n]^T\}| = 2+k+r,\\
&|\{i \in H : p_i = [3,1,2,4,\dots,n]^T\}| = 3+k+r,\\
&|\{i \in H : p_i = [3,2,1,4,\dots,n]^T\}| = 5+k,
\end{aligned}$$

and note that it has the same structure of those preference profiles described in (10). Then we have that

$$S_M^G(p) = \{[3,2,1,4,\dots,n]^T\}, \quad \text{and} \quad S_P^G(p) = \{[3,1,2,4,\dots,n]^T\},$$

so that $S_M^G(p) \cap S_P^G(p) = \varnothing$.

Statement 2. Consider at first $n = 4$. Since $\gcd(h, n!) = 1$ we have, in particular, $h \geq 5$ and h is odd. Note that $\frac{h-1}{2} \geq 2$ and $\frac{h+3}{2} \leq h-1$. Consider now $p \in \mathcal{P}$ such that, for every $i \in \{1,\dots,\frac{h-1}{2}\}$,

$$p_i = [1,2,3,4]^T, \quad p_{\frac{h-1}{2}+i} = [4,1,2,3]^T, \quad p_h = [3,4,1,2]^T. \tag{11}$$

First of all, note that there does not exist $(\varphi,\psi) \in S_h \times S_n$ such that $(\varphi,\psi,\rho_0) \in \mathrm{Stab}_G(p)$. Indeed, if by contradiction there is $(\varphi,\psi,\rho_0) \in \mathrm{Stab}_G(p)$, then it has to be $\psi[3,4,1,2]^T\rho_0 = [3,4,1,2]^T$ and then $\psi = (14)(23)$. Since

$$\psi[1,2,3,4]^T\rho_0 = [1,2,3,4]^T, \quad \psi[4,1,2,3]^T\rho_0 = [2,3,4,1]^T \neq [4,1,2,3]^T,$$

we have the contradiction. Then we have that $S_1^G(p) = \mathcal{L}(N)$. It is immediate to check that

$$C_{\frac{h+1}{2}}(p) = \varnothing, \quad C_{\frac{h+3}{2}}(p) = \left\{[1,2,3,4]^T, [1,2,4,3]^T, [1,4,2,3]^T, [4,1,2,3]^T\right\}.$$

Then $S_2^G(p) = C_{\frac{h+3}{2}}(p)$ and

$$S_M^G(p) = S_{MP}^G(p) = S_P^G(p) = S_{PM}^G(p) = \left\{[1,2,3,4]^T, [4,1,2,3]^T\right\}.$$

Assume now that $n \geq 5$. Since $\gcd(h, n!) = 1$ we have, in particular, $h \geq n+1$ and h is odd. Note that $\frac{h-1}{2} \geq 2$ and $\frac{h+3}{2} \leq h-1$. Consider now $p \in \mathcal{P}$ such that, for every $i \in \{1,\dots,\frac{h-1}{2}\}$,

$$p_i = [1,2,3,4,5,\dots,n]^T, \quad p_{\frac{h-1}{2}+i} = [4,1,2,3,5,\dots,n]^T, \quad p_h = [3,4,1,2,5,\dots,n]^T. \tag{12}$$

First of all, note that there does not exist $(\varphi,\psi) \in S_h \times S_n$ such that $(\varphi,\psi,\rho_0) \in \mathrm{Stab}_G(p)$. Indeed, assume by contradiction there is $(\varphi,\psi,\rho_0) \in \mathrm{Stab}_G(p)$. Consider then $p^* = p^{(id,id,\rho_0)} \in \mathcal{P}$. The preference profile p^* has the property that, for every $i \in H$, the top ranked alternative of p_i^* is n. As a consequence we have that $p = p^{(\varphi,\psi,\rho_0)} = p^{*(\varphi,\psi,id)} \in \mathcal{P}$ has the property that, for every $i \in H$, the top ranked alternative of p_i is $\psi(n)$. But that is a contradiction as the top ranked alternative of p_1 is 1 while the top ranked alternative of p_h is 3. Then we have that $S_1^G(p) = \mathcal{L}(N)$. It is immediate to check that

$$C_{\frac{h+1}{2}}(p) = \varnothing,$$

$$C_{\frac{h+3}{2}}(p) = \left\{[1,2,3,4,5,\dots,n]^T, [1,2,4,3,5,\dots,n]^T, [1,4,2,3,5,\dots,n]^T, [4,1,2,3,5,\dots,n]^T\right\}.$$

Then $S_2^G(p) = C_{\frac{h+3}{2}}(p)$ and

$$S_M^G(p) = S_{MP}^G(p) = S_P^G(p) = S_{PM}^G(p) = \left\{[1,2,3,4,5,\dots,n]^T, [4,1,2,3,5,\dots,n]^T\right\}.$$

8 The algorithm

When $n = 3$ and $\gcd(h, n!) = 1$, a careful analysis of the proof of Proposition 12 allows to compute the value of F^{MP} and F^{PM} on any preference profile. Given $p \in \mathcal{P}$, the algorithm to compute $F^{MP}(p)$ and $F^{PM}(p)$ is described below. In what follows, let $\psi_1 = id$, $\psi_2 = (23)$, $\psi_3 = (12)$, $\psi_4 = (123)$, $\psi_5 = (132)$, and $\psi_6 = (13)$.

Step 0.
Compute, for every $j \in \{1, \ldots, 6\}$, $v_j(p)$ as defined in (4). Choose $k \in \{1, \ldots, 6\}$ such that

$$v_k(p) = \max_{j \in \{1,\ldots,6\}} v_j(p).$$

Define $p^* = p^{(id, \psi_k^{-1}, id)}$ and compute $v_j(p^*)$ and, for every $x, y \in \{1, 2, 3\}$ with $x \neq y$, $s_{x,y}(p^*)$ as defined in (5). Write v_j instead of $v_j(p^*)$ and $s_{x,y}$ instead of $s_{x,y}(p^*)$. Note that $v_k(p) = v_1(p^*) \geq v_j(p^*)$ for all $j \in \{1, \ldots, 6\}$.

Step 1.
If $v_1 = v_5$, $v_3 = v_6$ and $s_{1,2} \leq s_{2,3}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[2, 3, 1]^T$.
If $v_1 = v_5$, $v_3 = v_6$ and $s_{1,2} > s_{2,3}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[1, 3, 2]^T$.
If $v_1 \neq v_5$ or $v_3 \neq v_6$ then go to Step 2.

Step 2.
If $v_2 = v_3$, $v_4 = v_5$ and $s_{3,1} \leq s_{1,2}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[1, 2, 3]^T$.
If $v_2 = v_3$, $v_4 = v_5$ and $s_{3,1} > s_{1,2}$ then $F^{MP}(p) = F^{PM}(p) = \psi_k[3, 2, 1]^T$.
If $v_2 \neq v_3$ or $v_4 \neq v_5$, then go to Step 3.

Step 3.
If $v_1 = v_4$, $v_2 = v_6$ and $s_{2,3} \leq s_{1,2}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[3, 1, 2]^T$.
If $v_1 = v_4$, $v_2 = v_6$ and $s_{2,3} > s_{1,2}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[2, 1, 3]^T$.
If $v_1 \neq v_4$ and $v_2 \neq v_6$, then go to Step 4.

Step 4.
If $s_{1,2} > h/2$, $s_{2,3} > h/2$ and $s_{3,1} < h/2$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[1, 2, 3]^T$.
If $s_{1,2} > h/2$, $s_{2,3} < h/2$ and $s_{3,1} > h/2$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[3, 1, 2]^T$.
If $s_{1,2} > h/2$, $s_{2,3} < h/2$ and $s_{3,1} < h/2$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[1, 3, 2]^T$.
If $s_{1,2} < h/2$, $s_{2,3} > h/2$ and $s_{3,1} > h/2$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[2, 3, 1]^T$.
If $s_{1,2} < h/2$, $s_{2,3} > h/2$ and $s_{3,1} < h/2$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[2, 1, 3]^T$.
If $s_{1,2} < h/2$, $s_{2,3} < h/2$ and $s_{3,1} > h/2$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[3, 2, 1]^T$.
If $s_{1,2} > h/2$, $s_{2,3} > h/2$ and $s_{3,1} > h/2$, go to Step 5.

Step 5.
If $s_{1,2}, s_{2,3} > s_{3,1}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[1, 2, 3]^T$.
If $s_{1,2}, s_{3,1} > s_{2,3}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[3, 1, 2]^T$.
If $s_{2,3}, s_{3,1} > s_{1,2}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[2, 3, 1]^T$.
If $s_{1,2} > s_{2,3} = s_{3,1}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[1, 2, 3]^T$.
If $s_{2,3} > s_{1,2} = s_{3,1}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[1, 2, 3]^T$.
If $s_{2,3} = s_{1,2} = s_{3,1}$, then $F^{MP}(p) = F^{PM}(p) = \psi_k[1, 2, 3]^T$.
If $s_{3,1} > s_{1,2} = s_{2,3}$, then go to Step 6.

Step 6.
If $v_4 > v_5$, then

$$F^{MP}(p) = \begin{cases} \psi_k[2, 3, 1]^T & \text{if } v_4 \geq v_6 \\ \psi_k[3, 2, 1]^T & \text{if } v_6 > v_4 \end{cases}, \quad F^{PM}(p) = \psi_k[2, 3, 1]^T$$

If $v_5 > v_4$, then

$$F^{MP}(p) = \begin{cases} \psi_k[3, 1, 2]^T & \text{if } v_5 \geq v_6 \\ \psi_k[3, 2, 1]^T & \text{if } v_6 > v_5 \end{cases}, \quad F^{PM}(p) = \psi_k[3, 1, 2]^T$$

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